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UNCONDITIONAL PROPERTIES OF ESTIMATORS OF REPEATED MEASUREMENTS MODEL

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ABSTRACT

We have four types of estimators of repeated measurements model and obtained these estimators by maximum likelihood method (MLM), restricted maximum likelihood method (REMLM) and modified restricted maximum likelihood method (MREMLM)(see [8] and [9]). In this paper, we study the unconditional properties of the repeated measures model of the four unbiased types.

INTRODUCTION

Repeated measurements are a concept that describes data in which the outcome variable is measured several times and under varying experimental conditions within each experimental device. Data from repeated measures is a typical type of multivariate data and associated error linear models that are commonly used in the simulation of data from repeated measures. Analysis of repeated measurements deals with reaction effects evaluated at various times or under different conditions on the same experimental unit. In many areas, such as health and life sciences, epidemiology, biomedical, environmental, manufacturing, psychological, educational studies, and so on, repeated measurement analysis is widely used. [7], [9], [11], [12], [13].

Many studies have explored the repeated measurements model. for example: Vonesh and Chinchilli (1997) discussed the univariate repeated measurements model, analysis of variance model,[12]. Al-Mouel (2004) studied the multivariate repeated measures models and comparison of estimators, [1]. Al-Mouel and Wang (2004) they studied the asymptotic expansion of the sphericity test for the one-way multivariate repeated measurements analysis of the variance model, [6]. AL-Mouel and Mustafa, in (2014) studied the sphericity test for one-way Multivariate Repeated Measurements Analysis of variance mode, [3] . AL-Mouel, and Naji, in (2014) devoted to study of one-way Multivariate repeated measurements analysis of covariance model, [4]. AL-Mouel and Hassan in (2016) estimate the repeated measurement model parameters by using maximum likelihood method, [5]. AL-Mouel and Kori in (2021) studied estimating the parameters of the repeated measurement model in two cases: conditional and unconditional, [2]. In this paper, we have five types of estimators of repeated measurements model, we obtained these estimators by maximum likelihood method (MLM), restricted maximum likelihood method (REMLM) and modified restricted maximum likelihood method (MREMLM). Our study focused on studying the unconditional properties of the four types of estimators.

Setting Up the Model

The repeated measurement model can be summarized as following:

$$h_{\text{abc}} = \theta + \text{A}_b + \pi_{a(b)} + \text{B}_c + (\text{AB})_{bc} + \epsilon_{abc} \quad (1)$$

where
$a = 1, \dots, I$ "is an index for experimental unit within group (b)",
$b = 1, \dots, J$ "is an index for levels of the between-units factor (Group)",
$c = 1, \dots, K$ "is an index for levels of the within-units factor (Time)",
h_{abc} : "is the response measurement at time (c) for unit (a) within group (b)",
θ : "is the overall mean",
A_b : "is the added effect for treatment group (b)",
$\pi_{a(b)}$:"is the random effect for due to experimental unit (a) within treatment group(b)",
B_c : "is the added effect for time (c)",
$(\text{AB})_{bc}$: "is the added effect for the group (b) $\times$ time (c) interaction",
ϵ_{abc}: "is the random error on time (c) for unit (a) within group (b)".
For the parameterization to be of full rank, we imposed the following set of conditions:

$\sum_{b=1}^{J} \text{A}_b = 0;\ \sum_{c=1}^{K} \text{B}_c = 0;\ \sum_{b=1}^{J} (\text{AB})_{bc} = 0$ for each c = 1,…, K;

$\sum_{c=1}^{K} (\text{AB})_{bc} = 0$ for each b = 1,…, J.

and let, the ϵ_{abc} and $\pi_{a(b)}$are independent with

$$\epsilon_{abc}\ i.i.d \sim N(0, \sigma_\epsilon^2) \text{ and } \pi_{a(b)}\ i.i.d \sim N(0, \sigma_\pi^2) \quad (2)$$

The sum of squares due to groups, subjects group, time, group $\times$ time and residuals are then defined respectively as follows:

$SS_A = IK\sum_{b=1}^{K}(\bar{h}_{.b.} - \bar{h}_{...})^2, SS_\pi = K\sum_{a=1}^{I}\sum_{b=1}^{J}(\bar{h}_{ab.} - \bar{h}_{.b.})^2$

$SS_B = IJ\sum_{c=1}^{K}(\bar{h}_{..c} - \bar{h}_{...})^2, SS_{A\times B} = I\sum_{b=1}^{J}\sum_{c=1}^{K}(\bar{h}_{.bc} - \bar{h}_{.b.} - \bar{h}_{..c} + \bar{h}_{...})^2$

$SS_\epsilon = \sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K}(\bar{h}_{abc} - \bar{h}_{.bc} - h_{ab.} + \bar{h}_{.b.})^2$

where
$\bar{h}_{...} = \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} h_{abc}$: the overall mean.
$\bar{h}_{.b.} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{c=1}^{K} y_{abc}$: the mean for group (b).
$\bar{h}_{ab.} = \frac{1}{K}\sum_{c=1}^{K} h_{abc}$: the mean for ath subject within group (b).
$\bar{h}_{..c} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} h_{abc}$: the mean for time (c).
$\bar{h}_{.bc} = \frac{1}{I}\sum_{a=1}^{I} h_{abc}$: the mean for group (b) at time (c).
Let

$$\theta_{abc} = \theta + \mathrm{A}_b + \pi_{a(b)} + \mathrm{B}_c + (\mathrm{AB})_{bc} \quad (3)$$

represent the mean of time (c) for unit (a) within group (b).
and, let

$$H = \ell_0\theta + \sum_{b=1}^{J}\ell_b A_b + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\pi_{a(b)} + \sum_{c=1}^{K}\ell_c B_c + \sum_{b=1}^{J}\sum_{c=1}^{k}\ell_b\ell_c(AB)_{bc} \quad (4)$$

an arbitrary linear combination of parameters θ, $\mathrm{A}_1, \dots, \mathrm{A}_q$, $\pi_{1(1)}, \dots, \pi_{I(J)}$, $\mathrm{B}_1, \dots, \mathrm{B}_K$, $(\mathrm{AB})_{11}, \dots, (\mathrm{AB})_{JK}$.
the best linear unbiased estimators (BLUE's) of the estimable parameters $\theta, A_b, \pi_{a(b)}, B_c, (AB)_{bc}$ and θ_{abc} are $\hat{\theta} = \bar{h}_{...}$, $\hat{A}_b = \bar{h}_{.b.} - \bar{h}_{...}$, $\hat{\pi}_{a(b)} = (1-\eta)(\bar{h}_{ab.} - \bar{h}_{.b.})$, $\hat{B}_c = \bar{h}_{..c} - \bar{h}_{...}$, $\left(\widehat{AB}\right)_{bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$ and $\hat{\theta}_{abc} = (1-\eta)\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right) + \bar{h}_{.bc}$,[2]
where (r) is the rate of expected mean squares, note that $0 < r \le 1$ is known iff $\sigma_\epsilon^2/\sigma_\pi^2$ is known.
By using maximum likelihood estimator (MLE) and related estimators and Jeffreys' noninformative prior and proper Bayes estimators, we have five kinds as follows: [8] and [9]

Type 1: This type consists of estimators as follows:

$\hat{\theta}_1 = \bar{h}_{...}$,
$\hat{\pi}_{1;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.})$,
$\hat{A}_{1;\,b} = \hat{A}_{1,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}$,
$\hat{B}_{1;\,c} = \hat{B}_{1,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}$,
$\left(\widehat{AB}\right)_{1;\,bc} = \left(\widehat{AB}\right)_{1,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}$,
$\hat{\theta}_{1;\,abc} = \hat{\theta}_{1,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{1,z;\,b} + \hat{\pi}_{1,z;\,a(b)} + \hat{\mathrm{B}}_{1,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{1,z;bc}$,
$\hat{\theta}_{1,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{1,z;\,a(b)}$,

And

$$\widehat{H}_1 = \ell_0\theta + \sum_{b=1}^{J} \ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.})\right] + \sum_{c=1}^{K} \ell_c\left(\bar{h}_{..c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

With

$$\hat{r}_1 = \hat{r}_{1,z} = z\,\frac{SS_\epsilon}{SS_\pi}.$$

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and z is an arbitrary positive constant. [8] and [9]

Type 2: This type consists of estimators as follows:

$$\begin{aligned}
&\hat{\theta}_2 = \bar{h}_{...}\,, \\
&\hat{\pi}_{2;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.}), \\
&\hat{A}_{2;\,b} = \hat{A}_{2,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}, \\
&\hat{B}_{2;\,c} = \hat{B}_{2,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}, \\
&\left(\widehat{AB}\right)_{2;\,bc} = \left(\widehat{AB}\right)_{2,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}, \\
&\hat{\theta}_{2;\,abc} = \hat{\theta}_{2,z;\,abc} = \bar{h}_{...} + \widehat{\mathrm{A}}_{2,z;\,b} + \hat{\pi}_{2,z;\,a(b)} + \widehat{\mathrm{B}}_{2,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{2,z;bc}, \\
&\hat{\theta}_{2,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{2,z;\,a(b)},
\end{aligned}$$

And

$$\widehat{H}_1 = \ell_0\theta + \sum_{b=1}^{J} \ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J} \ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.})\right] + \sum_{c=1}^{K} \ell_c\left(\bar{h}_{..c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K} \ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

With

$$\hat{r}_2 = \hat{r}_{2,z} = \min\{z\,\frac{SS_\epsilon}{SS_\pi}, 1\},$$

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and z is an arbitrary positive constant. [8] and [9]

Type 3: This type consists of estimators as follows:

$$\begin{aligned}
&\hat{\theta}_3 = \bar{h}_{...}\,, \\
&\hat{\pi}_{3;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.}), \\
&\hat{A}_{3;\,b} = \hat{A}_{3,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...}, \\
&\hat{B}_{3;\,c} = \hat{B}_{3,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...}, \\
&\left(\widehat{AB}\right)_{3;\,bc} = \left(\widehat{AB}\right)_{3,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}, \\
&\hat{\theta}_{3;\,abc} = \hat{\theta}_{3,z;\,abc} = \bar{h}_{...} + \widehat{\mathrm{A}}_{3,z;\,b} + \hat{\pi}_{3,z;\,a(b)} + \widehat{\mathrm{B}}_{3,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{3,z;bc}, \\
&\hat{\theta}_{3,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{3,z;\,a(b)},
\end{aligned}$$

And

$$\widehat{H}_3 = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.})\right] + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

With

$\hat{r}_3 = \hat{r}_{3,z} = \mathrm{f}_3\ (SS_\pi, SS_\epsilon),$

where $(a = 1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$ and $\mathrm{f}_3(x, y)$ is an arbitrary function of $x, y > 0$. [8] and [9]

<u>Type 4</u>: This type consists of the following estimators:

$\hat{\theta}_4 = \bar{h}_{...}$,
$\hat{\pi}_{4;\,a(b)} = (1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.}),$
$\hat{A}_{4;\,b} = \hat{A}_{4,z\,;\,b} = \bar{h}_{.b.} - \bar{h}_{...},$
$\hat{B}_{4;\,c} = \hat{B}_{4,z\,;\,c} = \bar{h}_{..c} - \bar{h}_{...},$
$\left(\widehat{AB}\right)_{4;\,bc} = \left(\widehat{AB}\right)_{4,z;\,bc} = \bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c},$
$\hat{\theta}_{4;\,abc} = \hat{\theta}_{4,z;\,abc} = \bar{h}_{...} + \hat{\mathrm{A}}_{4,z;\,b} + \hat{\pi}_{4,z;\,a(b)} + \hat{\mathrm{B}}_{4,z;\,c} + \left(\widehat{\mathrm{AB}}\right)_{4,z;bc},$
$\hat{\theta}_{4,z;\,abc} = \bar{h}_{.bc} + \hat{\pi}_{4,z;\,a(b)},$

And

$$\widehat{H}_4 = \ell_0\theta + \sum_{b=1}^{J}\ell_b\left(\bar{h}_{.b.} - \bar{h}_{...}\right) + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_b\ell_c\left[(1-\hat{r})(\bar{h}_{ab.} - \bar{h}_{.b.})\right] + \sum_{c=1}^{K}\ell_c\left(\bar{h}_{..c} - \bar{h}_{...}\right) + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c\left(\bar{h}_{.bc} + \bar{h}_{...} - \bar{h}_{.b.} - \bar{h}_{..c}\right),$$

With

$\hat{r}_4 = \hat{r}_4, z = \mathrm{f}_4\ \left(\frac{SS_\epsilon}{SS_\pi}\right),$

where $(i = 1, \dots, n, j = 1, \dots, q, k = 1, \dots, q)$ and $\mathrm{f}_4(x_1, x_2)$ is an arbitrary positive function of $x_1 > 0$ and $x_2 > 0$. [8] and [9]

Unconditional Properties of Type 4 And 3 Estimators

<u>Theorem 1</u>: For model (2.1), let $\hat{r}_4$ be an arbitrary Type 4 estimator of r and let $\hat{\theta}_4, \hat{A}_{4;\,b}, \hat{\pi}_{4;\,a(b)}$, $\hat{B}_{4;\,c}$ and $\left(\widehat{AB}\right)_{4;\,bc}$ be the corresponding Type 4 estimators of $\theta, A_b, \pi_{a(b)}, B_c\ and\ (AB)_{bc}$ respectively, $(a = 1, \dots, I;\ b = 1, \dots, J, c = 1, \dots, K)$. If the expectations of $\hat{\theta}_4, \hat{A}_{4;\,b}, \hat{\pi}_{4;\,a(b)}$, $\hat{B}_{4;\,c}$ and $\left(\widehat{AB}\right)_{4;\,bc}$ exists, then $\hat{\theta}_4, \hat{A}_{4;\,b}, \hat{\pi}_{4;\,a(b)}$, $\hat{B}_{4;\,c}$ and $\left(\widehat{AB}\right)_{4;\,bc}$ estimates $\theta, A_b, \pi_{a(b)}, B_c\ and\ (AB)_{bc}$ respectively are unbiased.

Proof:

Since $\hat{\theta}_4 = \bar{h}_{...} \rightarrow E(\hat{\theta}_4) = E(\bar{h}_{...})$ and since $E(\bar{h}_{...}) = E(\frac{1}{IJK}\sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K} h_{abc})$

, then $E(\hat{\theta}_4) = E(\frac{1}{IJK}\sum_{a=1}^{I}\sum_{b=1}^{J}\sum_{c=1}^{K} h_{abc})$,

by assumptions of this model we get

$E(\hat{\theta}_4) = \frac{1}{IJK}(IJK\ \theta) = \theta$, we get $\hat{\theta}_4$ is unbiased estimator of θ,
Since $\hat{A}_{4;\,b} = \bar{h}_{.b.} - \bar{h}_{...}, \rightarrow E(\hat{A}_{4;\,b}) = E(\bar{h}_{.b.} - \bar{h}_{...}) = E(\bar{h}_{.b.}) - E(\bar{h}_{...})$ and $E(\hat{\theta}) = \bar{h}_{...}$, we get
$E(\hat{A}_{4;\,b}) = E(\bar{h}_{.b.}) - \theta = E\left(\frac{1}{IK}\sum_{a=1}^{I}\sum_{c=1}^{K} h_{abc}\right) - \theta$

by assumptions of this model we get

$E(\hat{A}_{4;\,b}) = \frac{1}{IK}(IK\theta + IK\mathrm{A}_b) - \theta \rightarrow E(\hat{A}_{4;\,b}) = \mathrm{A}_b$, we get $\hat{A}_{4;\,b}$ is unbiased estimator of A_b,

In the same way, it can be concluded that the remaining parameters

$E(\hat{\pi}_{4;\,a(b)}) = E(\pi_{a(b)})$, $E(\hat{B}_{4;\,c}) = \mathrm{B}_c$ and $E\left((\widehat{AB})_{4;\,bc}\right) = (\mathrm{AB})_{bc}$ are unbiased.

<u>**Corollary 1**</u>: For model (1), Let $\hat{r}_4$ be an arbitrary Type 4 estimator of r, and let $\hat{H}_4$ and $\hat{\pi}_{4;1(1)}$ be the corresponding type 4 estimators of H and $\pi_{1(1)}$. Then, if $E(\hat{\pi}_{4;1(1)})$ exsists, $\hat{H}_4$estimators H unbiaseedly.

Proof:

$$\hat{H}_4 = \ell_0\hat{\theta} + \sum_{b=1}^{J}\ell_b\hat{A}_{4,z;\,b} + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\hat{\pi}_{4,z\,;\,a(b)} + \sum_{c=1}^{K}\ell_c\hat{B}_{4,z;\,c} + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_k(\widehat{AB})_{4,z;bc}$$

$$E(\hat{H}_4) = E(\ell_0\hat{\theta} + \sum_{b=1}^{J}\ell_b\hat{A}_{4,z;\,b} + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\hat{\pi}_{4,z\,;\,a(b)} + \sum_{c=1}^{K}\ell_c\hat{B}_{4,z;\,c} + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c(\widehat{AB})_{4,z;bc})$$

$$= \ell_0\hat{\theta} + \sum_{b=1}^{J}\ell_b A_{4,z;\,b} + \sum_{a=1}^{I}\sum_{b=1}^{J}\ell_a\ell_b\,\pi_{4,z\,;\,a(b)} + \sum_{c=1}^{K}\ell_c B_{4,z;\,c} + \sum_{b=1}^{J}\sum_{c=1}^{K}\ell_b\ell_c(AB)_{4,z;bc} = H$$

which include that the estimators of H in corollary (3.1) of type 1, 2, 3 and 4 are unbiased.

<u>**Lemma 3.1**</u>: Let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\hat{\pi}_{4;a(b)}$ be the corresponding type 4 estimators of $\pi_{a(b)}$ $(a = 1, \ldots, I; b = 1, \ldots, J)$. Then

under model (1),

$$E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=-\frac{\sigma_{\pi}^{2}}{IJ}=-\frac{\sigma_{\epsilon}^{2}(1-r)}{IJK} \quad (5)$$

Proof:

$$E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=E[\left(\bar{h}_{\dots}-\theta\right)\hat{\pi}_{4;a(b)}-\left(\bar{h}_{\dots}-\theta\right)\left(-\pi_{a(b)}\right)]$$

by symmetry we get

$$E\left[\left(\bar{h}_{\dots}-\theta\right)\hat{\pi}_{4;a(b)}\right]=\frac{1}{IJ}E\left[\left(\bar{h}_{\dots}-\theta\right)\sum_{a}\sum_{b}\hat{\pi}_{4;a(b)}\right]=0$$

and hence

$$E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=-E\left[\left(\bar{h}_{\dots}-\theta\right)\pi_{a(b)}\right]=-E[\left(\theta+\bar{A}_{.}+\bar{\pi}_{.(.)}+\bar{B}_{.}+(\overline{AB}_{.})+\bar{\epsilon}_{\dots}-\theta\right)\pi_{a(b)}]=$$

$$-E([\left(\theta+\frac{1}{J}\sum_{b=1}^{J}A_{b}+\frac{1}{IJ}\sum_{a=1}^{I}\sum_{b=1}^{J}\pi_{a(b)}+\frac{1}{K}\sum_{c=1}^{K}B_{c}+\frac{1}{JK}\sum_{b=1}^{J}\sum_{c=1}^{K}(AB)_{bc}+\bar{\epsilon}_{\dots}-\theta\right)\pi_{a(b)}]$$

by assumption of model (1) we have,

$$E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=-E\left[\left(\bar{\pi}_{.(.)}+\epsilon_{\dots}\right)\pi_{a(b)}\right]=-E[\bar{\pi}_{.(.)}\pi_{a(b)}]$$

$$\therefore E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=-\frac{\sigma_{\pi}^{2}}{IJ}$$

and

$$-\frac{\sigma_{\epsilon}^{2}(1-r)}{rIJK}=-\frac{\sigma_{\epsilon}^{2}\left(1-\frac{\sigma_{\epsilon}^{2}}{K\sigma_{\pi}^{2}+\sigma_{\epsilon}^{2}}\right)}{\frac{IJK\sigma_{\epsilon}^{2}}{K\sigma_{\pi}^{2}+\sigma_{\epsilon}^{2}}}=-\frac{\frac{K\sigma_{\pi}^{2}}{K\sigma_{\pi}^{2}+\sigma_{\epsilon}^{2}}}{\frac{IJK}{K\sigma_{\pi}^{2}+\sigma_{\epsilon}^{2}}}=-\frac{K\sigma_{\pi}^{2}}{IJK}=-\frac{\sigma_{\pi}^{2}}{IJ}$$

$$\therefore E\left[\left(\bar{h}_{\dots}-\theta\right)\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\right]=-\frac{\sigma_{\pi}^{2}}{IJ}=-\frac{\sigma_{\epsilon}^{2}(1-r)}{rIJK}$$

<u>Lemma 2</u>: Let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\hat{\pi}_{4;a(b)}$ be the corresponding type 4 estimators of $\pi_{a(b)}$ $(a=1,\dots,I;b=1,\dots J)$. Then under the model (1),

$$E\left[\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\left(\hat{\pi}_{4;a'(b')}-\pi_{a'(b')}\right)\right]=(\frac{1}{(I-1)(J-1)}[\frac{\sigma_{\epsilon}^{2}(1-r)}{rIK}-MSE(\hat{\pi}_{4;1(1)}-\pi_{1(1)}]) \quad ,(a>a'=1,\dots,I;b>b'=1,\dots J). \quad (6)$$

<u>Proof:</u> Let $\hat{r}=\hat{r}_4$ and $\hat{\pi}_{a(b)}=\hat{\pi}_{4;a(b)}$ by symmetry, we get

$$E\left[\left(\hat{\pi}_{a(b)}-\pi_{a(b)}\right)\left(\hat{\pi}_{a'(b')}-\pi_{a'(b')}\right)\right]=\frac{1}{(I-1)(J-1)}\sum_{\substack{a=1\\a\neq a'}}\sum_{\substack{b=1\\b\neq b'}}E\left[\left(\hat{\pi}_{a(b)}-\pi_{a(b)}\right)\left(\hat{\pi}_{a'(b')}-\pi_{a'(b')}\right)\right]$$

$$=\frac{1}{(I-1)(J-1)}\sum_{a=1}^{n}E\left[\left(\hat{\pi}_{a(b)}-\pi_{a(b)}\right)\left(\hat{\pi}_{a'(b')}-\pi_{a'(b')}\right)\right]-$$

$\frac{1}{(I-1)(J-1)}E\left(\hat{\pi}_{a'(b')}-\pi_{a'(b')}\right)^2$
$=\frac{1}{(I-1)(J-1)}\{E[(1-\hat{r})(\bar{h}_{ab.}-\bar{h}_{.b.})(-\sum_a\sum_b\pi_{a(b)})][-\sum_a\sum_b\pi_{a(b)}]+\sigma_\pi^2-MSE(\hat{\pi}_{1(1)}-\pi_{1(1)}\}$

by symmetry, we have

$E[(1-\hat{r})(\bar{h}_{ab.}-\bar{h}_{.b.})(-\sum_a\sum_b\pi_{a(b)})]=-\frac{1}{IJ}\sum_a\sum_b E[\Big((1-\hat{r})(\bar{h}_{ab.}-\bar{h}_{.b.})(\sum_a\sum_b\pi_{a(b)})\Big)=0.$
$=\sigma_\pi^2-MSE(\hat{\pi}_{1(1)}-\pi_{1(1)})$

since $\frac{\sigma_\pi^2}{IJ}=\frac{\sigma_\epsilon^2(1-r)}{rIJK}$, we get

$E\left[\left(\hat{\pi}_{4;a(b)}-\pi_{a(b)}\right)\left(\hat{\pi}_{4;a'(b')}-\pi_{a'(b')}\right)\right]=(\frac{1}{(I-1)(J-1)}[\frac{\sigma_\epsilon^2(1-r)}{rIK}-MSE(\hat{\pi}_{4;1(1)}-\pi_{1(1)}])\quad,(a>a'=1,\ldots,I;b>b'=1,\ldots J).$

<u>Theorem 2</u>: For model (1), let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\widehat{H}_4$ and $\hat{\pi}_{4;1(1)}$ be the corresponding type 4 estimators of H and $\pi_{1(1)}$. Then,

$$MSE(\widehat{H}_4,H)=\frac{\sigma_\epsilon^2}{rIJK}\left[\ell_0^2+\sum_{b=1}^J\ell_b^2(I-1)+\sum_{a=1}^I\sum_{b=1}^J\ell_a^2\ell_b^2[(1-\hat{r}_4)^2(I-1)(1-IJ)]+\sum_{c=1}^J r\ell_c^2(K-1)+\sum_{b=1}^J\sum_{c=1}^K r\ell_j^2\ell_c^2J(K-1)(J-1)\right]\quad(7)$$

<u>Proof:</u> let $A_b=A_{4;b},B_c=B_{4,c},\hat{\pi}_{a(b)}=\hat{\pi}_{4\,;\,a(b)}$ and $(AB)_{bc}=(AB)_{4;\,bc}$we get

$MSE(\widehat{H}_4,H)=Var(\widehat{H}_4-h)=$
$Var\Big[\Big(\ell_0\bar{h}_{...}+\sum_{b=1}^J\ell_j\hat{A}_{4,z;\,b}+\sum_{a=1}^I\sum_{b=1}^J\ell_a\ell_b\,\hat{\pi}_{4,z\,;a(b)}+\sum_{c=1}^K\ell_c\hat{B}_{4,z;\,c}+\sum_{b=1}^J\sum_{c=1}^K\ell_b\ell_c(\widehat{AB})_{4,z;\,bc}\Big)-\Big(\ell_0\theta+\sum_{b=1}^J\ell_bA_b+\sum_{a=1}^I\sum_{b=1}^J\ell_a\ell_b\,\pi_{a(b)}+\sum_{c=1}^K\ell_cB_c+\sum_{b=1}^J\sum_{c=1}^K\ell_b\ell_c(AB)_{bc}\Big)\Big]$
$=Var[\ell_0(\hat{\theta}-\theta)+\sum_{b=1}^J\ell_b(\hat{A}_{4;\,b}-A_b)+\sum_{a=1}^I\sum_{b=1}^J\ell_a\ell_b(\hat{\pi}_{4\,;\,a(b)}-\pi_{a(b)})+\sum_{c=1}^k\ell_c(\hat{B}_{4;\,c}-B_c)+\sum_{b=1}^J\sum_{c=1}^K\ell_b\ell_c((AB)_{4;\,bc}-(AB)_{bc})\,]$
$=Var[\ell_0(\bar{h}_{...}-\theta)+\sum_{b=1}^J\ell_b(\bar{h}_{.b.}-h_{...}-A_b)+\sum_{a=1}^I\sum_{b=1}^J\ell_a\ell_b\big((1-\hat{r}_{1,z})(\bar{h}_{bc.}-\bar{h}_{.b.})-\pi_{a(b)}\big)+\sum_{c=1}^K\ell_c(\bar{h}_{..c}-\bar{h}_{...}-B_c)+\sum_{b=1}^J\sum_{c=1}^K\ell_b\ell_c(\bar{h}_{.bc}+\bar{h}_{...}-h_{.b.}-\bar{h}_{..c}-(AB)_{bc})\,]$
$=\frac{\sigma_\epsilon^2}{rIJK}[\ell_0^2+\sum_{b=1}^J\ell_b^2(I-1)+\sum_{a=1}^I\sum_{b=1}^J\ell_a^2\ell_b^2[(1-\hat{r}_4)^2(I-1)(1-IJ)]+\sum_{c=1}^J r\ell_c^2(K-1)+\sum_{b=1}^J\sum_{c=1}^K r\ell_b^2\ell_c^2J(K-1)(J-1)\,]$

<u>Theorem 3</u>: Let $\hat{r}_4$ be an arbitrary type 4 estimator of r. Then under the model (1), $[(1-r)(\bar{h}_{ab.}-\bar{h}_{.b.})-\pi_{\,a(b)}]$ is distributed independently of $[(\hat{r}_4-r)(\bar{h}_{a'b'.}-\bar{h}_{.b'.})]$ $(a,a'=1,\ldots,I;b,b'=1,\ldots J)$, and $E\{[(1-$

$$r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}\right]\left[(\hat{r}_4-r)\left(\bar{h}_{a'b'.}-\bar{h}_{.b'.}\right)\right]\} = 0$$
$$(a,a'=1,\dots,I;b,b'=1,\dots J). \qquad (8)$$

Proof: The quantity $(\hat{r}_4-r)\left(\bar{h}_{a'b'.}-\bar{h}_{.b'.}\right)$ is a function of the error contrasts $(\epsilon_{abc}=h_{abc}-\bar{h}_{...})$. Therefore, it suffices to prove that $\left((1-\hat{r})\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}\right)$ is distributed independently of the error contrasts.

Let $d_{ab}, d_{a'b'} = \frac{1}{(I-1)(J-1)}, if\ a,b=a'b',$
$= 1, \quad if\ a,b \neq a',b'.$

we have that

$$E\{\left[(1-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}\right]\left[\left(\bar{h}_{a'b'.}-\bar{h}_{.b'.}\right)\right]\}$$
$$= E\{[(1-r)(\theta+A_b+\pi_{a(b)}+B_c+(\overline{AB})_{b.}+\bar{\epsilon}_{ab.}-\theta-A_b-\bar{\pi}_{.(b)}-\bar{B}_{.}$$
$$-(\overline{AB})_{b.}-\bar{\epsilon}_{.b.})-\pi_{a(b)}][\theta+A_{b'}+\pi_{a'(b')}+B_c+(\overline{AB})_{b'.}$$
$$+\bar{\epsilon}_{a'b'.}-\theta-A_{b'}-\bar{\pi}_{.(b')}-\bar{B}_{.}-(\overline{AB})_{b'.}-\bar{\epsilon}_{.b'.}]\}$$
$$= d_{ab}, d_{a'b'}E\{[(1-r)\left(\pi_{a(b)}-\bar{\pi}_{.(b)}+\bar{\epsilon}_{ab.}-\bar{\epsilon}_{.b.}\right)-\pi_{a(b)}][\pi_{a'(b')}-$$
$$\bar{\pi}_{.(b')}+\bar{\epsilon}_{a'b'.}-\bar{\epsilon}_{.b'.}]$$
$$= d_{ab}, d_{a'b'}\left\{(1-r)\frac{(I-1)\sigma_\epsilon^2}{rIK}-(1-r)\frac{(I-1)\sigma_\epsilon^2}{rIK}\right\}=0$$

since $\left[(1-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}\right]$ and $\left[\left(\bar{h}_{a'b'.}-\bar{h}_{.b'.}\right)\right]$ are normally distributed and are uncorrelated, they are distributed independently. Then result (8) are independently distribution.

Theorem 4: Let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\hat{\pi}_{4;a(b)}$ be the corresponding type 4 estimators of $\pi_{a(b)}$. Then under the model (1),

$$MSE\left(\hat{\pi}_{4;a(b)},\pi_{a(b)}\right) = MSE\left(\hat{\pi}_{a(b)},\pi_{a(b)}\right) + \frac{1}{IJK}E[(\hat{r}_4-r)^2SS_\pi]$$
$$= \frac{\sigma_\epsilon^2(1-r)}{rIK}+\frac{1}{IK}E[(\hat{r}_4-r)^2SS_\pi] \qquad (9)$$

Proof:

$$MSE\left(\hat{\pi}_{4;a(b)},\pi_{a(b)}\right) = E[\hat{\pi}_{4;a(b)}-\pi_{a(b)}]^2 = var[(1-\hat{r}_4)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-$$
$$\pi_{a(b)}]$$
$$= var[(1-\hat{r}_4+r-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}]$$
$$= var[(1-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)+(\hat{r}_4-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)-\pi_{a(b)}]$$

since $\hat{\pi}_{a(b)} = [(1-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)$, then

$$= var[\left(\hat{\pi}_{a(b)}-\pi_{a(b)}\right)^2+(\hat{r}_4-r)\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)] \qquad (10)$$

by symmetry we get

$$= MSE\left(\hat{\pi}_{a(b)},\pi_{a(b)}\right)+\frac{K}{IJK}\sum_{a=1}^{I}\sum_{b=1}^{J}E[(\hat{r}_4-r)^2\left(\bar{h}_{ab.}-\bar{h}_{.b.}\right)^2]$$

$= MSE(\hat{\pi}_{a(b)}, \pi_{a(b)}) + \frac{1}{IJK} E[(\hat{r}_4 - r)^2 SS_\pi]$

If r is unknown, then the second term $\frac{1}{IJK} E[(\hat{r}_4 - r)^2 SS_\pi]$ is a penalty term.

Theorem 5: Let $f_3\left(\frac{SS_\epsilon}{SS_\pi}\right)$ where $f_3(t)$ is an arbitrary positive function of $t > 0$. Then, under the model (1),

$E[(\hat{r}_3 - r)^2 SS_\pi] = \frac{\sigma_\epsilon^2 JK(I-1)}{r} E[f_3(\frac{rx}{1-x} - r)^2(1-x)]$, where x is a beta random variable with parameters $\frac{1}{2}J(I-1)$ and $\frac{1}{2}J(K-1)(I-1)$.

Proof: Let $v_\epsilon = \frac{1}{2}J(I-1)$, $v_\pi = \frac{1}{2}J(K-1)(I-1)$, $u_\epsilon = \frac{SS_\epsilon}{\tau_\epsilon}, u_\pi = \frac{SS_\pi}{\tau_\pi}, \varpi = u_\epsilon + u_\pi$ and $x = \frac{u_\epsilon}{u_\epsilon + u_\pi}$. For all random variables u_ϵ and u_π are distributed independently as a chi-square random variable with degrees of freedom $J(I-1)$ and $J(K-1)(I-1)$, respectively. And thus, ϖ has a chi-square distribution with degree of freedom $JK(I-1)$, x has a beta random variable with parameters $\frac{1}{2}J(I-1)$ and $\frac{1}{2}J(K-1)(I-1)$. And ϖ and x are distributed independently.

therefore,

$\frac{SS_\epsilon}{SS_\pi} = \frac{\tau_\epsilon u_\epsilon}{\tau_\pi u_\pi} \cdot \frac{u_\epsilon + u_\pi}{u_\epsilon + u_\pi} = r\left(\frac{u_\epsilon}{u_\epsilon + u_\pi}\right)\left(\frac{u_\epsilon + u_\pi}{u_\pi}\right)$
where $\frac{1}{(1-x)} = \left(\frac{u_\epsilon + u_\pi}{u_\pi}\right)$ and $x = \frac{u_\epsilon}{u_\epsilon + u_\pi}$, then
$\frac{SS_\epsilon}{SS_\pi} = \frac{rx}{(1-x)} \to \frac{SS_\pi}{SS_\epsilon} = \frac{(1-x)}{rx} \to SS_\pi = \frac{\tau_\epsilon u_\epsilon (1-x)}{\frac{\tau_\epsilon}{\tau_\pi}\left(\frac{u_\epsilon}{u_\epsilon + u_\pi}\right)} = \tau_\pi (1-x)\varpi = \frac{\sigma_\epsilon^2 \varpi (1-x)}{r}$

thus,

$E[(\hat{r}_3 - r)^2 SS_\pi] = E[f_3(\frac{rx}{1-x} - r)^2 \tau_\pi (1-x)\varpi]$
$= \tau_\pi E(\varpi) E\{[f_3\left(\frac{rx}{1-x} - r)^2 \tau_\pi (1-x)\right]\}$

and,

$\tau_\pi E(\varpi) = \tau_\pi E(u_\epsilon + u_\pi) = \tau_\pi E\left[\frac{SS_\epsilon}{\tau_\epsilon} + \frac{SS_\pi}{\tau_\pi}\right] = \tau_\pi \left[\frac{\tau_\epsilon}{\tau_\epsilon} + \frac{SS_\pi}{\tau_\pi}\right]$
$= \tau_\pi \left[\frac{\tau_\epsilon J(K-1)(I-1)}{\tau_\epsilon} + \frac{\tau_\pi J(I-1)}{\tau_\pi}\right] = (K\sigma_\pi^2 + \sigma_\epsilon^2) JK(I-1)$
$= \left(\frac{IJK\sigma_\epsilon^2(1-r)}{rIJK} + \sigma_\epsilon^2\right) JK(I-1) = \left(\frac{\sigma_\epsilon^2(1-r) + r\sigma_\epsilon^2}{r}\right) JK(I-1)$
$\tau_\pi E(\varpi) = \frac{\sigma_\epsilon^2 JK(I-1)}{r}$, then we have
$E[(\hat{r}_3 - r)^2 SS_\pi] = \frac{\sigma_\epsilon^2 JK(I-1)}{r} E[f_3(\frac{rx}{1-x} - r)^2(1-x)]$

Theorem 6: For model (1), let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\widehat{\boldsymbol{\theta}}_4$ and $\widehat{\boldsymbol{\theta}}_{4;abc}$ be the corresponding Type 4 estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_{abc}$ ($a =$

$1, \dots, I;\ b = 1, \dots, J;\ c = 1, \dots, K)$. Then, if $E(\theta_{4;111})$ exsists,

$$TB(\widehat{\boldsymbol{\theta}}_4, \boldsymbol{\theta}) = \mathbf{0} \tag{11}$$

where $\boldsymbol{\theta} = (\theta_{111}, \dots, \theta_{IJK})^T$.

Theorem 7: For model (1), let $\hat{r}_4$ be an arbitrary type 4 estimator of r, and let $\widehat{\boldsymbol{\theta}}_4, \hat{\theta}_{4;abc}$ and $\hat{\pi}_{4;a(b)}$ be the corresponding type 4 estimators of $\boldsymbol{\theta}$, θ_{abc} and $\boldsymbol{\pi}_{4;a(b)}$ $(a = 1, \dots, I;\ b = 1, \dots J;\ c = 1, \dots, K)$. Then,

(i) $TMSE(\widehat{\boldsymbol{\theta}}_4, \boldsymbol{\theta}) = IJK \sum_{a=1}^{I} \sum_{b=1}^{J} \sum_{c=1}^{K} E\left(\hat{\theta}_{4;abc} - \theta_{4;abc}\right)^2$
$= IJK\ MSE(\hat{\theta}_{4;abc}, \theta_{4;abc})$ (12)

(ii) $MSE\left(\hat{\theta}_{4;abc}, \theta_{abc}\right) = MSE\left(\hat{\pi}_{4;1(1)}, \pi_{1(1)}\right) + \frac{\sigma_\epsilon^2[r(IK-1)-1]}{rI^2K}$ (13)

Proof: (ii)

$MSE\left(\hat{\theta}_{4;abc}, \theta_{abc}\right) = E[\hat{\theta}_{4;abc} - \vartheta_{abc}]^2 = E[\bar{h}_{.bc} + \hat{\pi}_{4\,;\,a(b)} - \vartheta_{abc}]^2$
$= E[\theta + A_b + \bar{\pi}_{.(b)} + B_c + (AB)_{bc} + \bar{\epsilon}_{.bc} + \hat{\pi}_{4\,;\,a(b)} - \big(\theta + A_b + \pi_{a(b)} + B_c + (AB)_{bc}\big)]^2$
$= E[(\bar{\pi}_{.(b)} + \bar{\epsilon}_{.bc})^2 + 2\left(\bar{\pi}_{.(b)} + \bar{\epsilon}_{.bc}\right)\left(\hat{\pi}_{4\,;\,a(b)} - \pi_{a(b)}\right) + \left(\hat{\pi}_{4\,;\,a(b)} - \pi_{a(b)}\right)^2]$
Since $E[\left(\hat{\pi}_{4\,;\,a(b)} - \pi_{a(b)}\right)^2 = MSE(\hat{\pi}_{4\,;\,a(b)}, \pi_{a(b)})$, we get
$= \frac{\sigma_\epsilon^2[r(IK-1)-1]}{rI^2K} + MSE\left(\hat{\pi}_{4\,;\,a(b)}, \pi_{a(b)}\right)$,then
$MSE\left(\hat{\theta}_{4;abc}, \theta_{4;abc}\right) = MSE\left(\hat{\pi}_{4;1(1)}, \pi_{1(1)}\right) + \frac{\sigma_\epsilon^2[r(IK-1)-1]}{rI^2K}$

Unconditional Properties of Type 1 Estimators

Theorem 8: For model (1), let $\hat{r}_1$ be an arbitrary type 1 estimator of r, and let $\hat{\pi}_{1;a(b)}$ be the corresponding type 4 estimators of $\pi_{1(1)}$ $(a = 1, \dots, I; b = 1, \dots, J)$. Then,

If $I > 3, E\left[\left|\hat{\pi}_{1;a(b)}\right|\right] < \infty$.

Proof: For any random variable x, we have that

$$E[|x|] \leq [E(x^2)]^{\frac{1}{2}} \tag{14}$$

Inequality (14) is a special case of Holder's inequality. Note that

$\left[\left|\hat{\pi}_{1;a(b)}\right|\right] = E[|1 - \hat{r}_1|.\left|\bar{h}_{ab.} - \bar{h}_{.b.}\right|] < \infty$
iff $E[\hat{r}_1\left|\bar{h}_{ab.} - \bar{h}_{.b.}\right|] < \infty$. Thus, using (14),
$E[\hat{r}_1\left|\bar{h}_{ab.} - \bar{h}_{.b.}\right|] \leq \left\{E\left[r_1^2\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right)^2\right]\right\}^{\frac{1}{2}}$
$= \frac{K}{IJK} \sum_{a=1}^{I} \sum_{b=1}^{J} \left\{E\left[r_1^2\left(\bar{h}_{ab.} - \bar{h}_{.b.}\right)^2\right]\right\}^{\frac{1}{2}} = \frac{1}{IJK}\{E[r_1^2 SS_\pi]\}^{\frac{1}{2}}$

$$= \frac{1}{IJK}\left\{E\left[\left(\frac{zSS_\epsilon}{SS_\pi}\right)^2 SS_\pi\right]\right\}^{\frac{1}{2}} = \frac{1}{IJK}\left\{E\left[\frac{z^2SS_\epsilon^2}{SS_\pi}\right]\right\}^{\frac{1}{2}}$$
$$= \frac{1}{IJK}\left\{\left(\frac{\tau_\epsilon^2}{\tau_\pi}\right)\frac{z^2J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)}\right\} < \infty. I > 3. \qquad (15)$$

Theorem 9: For model (1), let z be an arbitrary positive constant and let $\pi_{1,z;(a)b}$ be the corresponding type 1 estimators of $\pi_{(a)b}(a = 1, \dots, I; b = 1, \dots, J)$. Then,

$$MSE(\hat{\pi}_{1,z;(a)b}, \pi_{(a)b}) = \frac{\sigma_\epsilon^2}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)] + 1\right\} + \frac{\sigma_\epsilon^2 r}{IJK}\left\{\frac{z^2J(K-1)(I-1)(IJK-JK-IK-J+2)}{J(I-3)} - 2zJ(K-1)(I-1) + J(I-1)\right\}$$
(16)

Proof: : For model (1), let $\hat{r} = \hat{r}_{1,k}$ and $\hat{\pi}_{(a)b}$ be the corresponding untruncated estimators of r and $\pi_{1,z;(a)b}$ respectively, we get

$MSE(\hat{\pi}_{1;a(b)}, \pi_{a(b)}) = E[\hat{\pi}_{1;a(b)} - \pi_{a(b)}]^2$
$= E[(1 - \hat{r} + r - r)(\bar{h}_{ab.} - \bar{h}_{.b.}) - \pi_{a(b)}]^2$
$= E[(1 - r)(\bar{h}_{ab.} - \bar{h}_{.b.}) + (\hat{r} - r)(\bar{h}_{ab.} - \bar{h}_{.b.}) - \pi_{a(b)}]^2$
since $\hat{\pi}_{a(b)} = [(1 - r)(\bar{h}_{ab.} - \bar{h}_{.b.})$, then
$= E[(\hat{\pi}_{a(b)} - \pi_{a(b)})^2 + (\hat{r} - r)^2(\bar{h}_{ab.} - \bar{h}_{.b.})^2]$
since $MSE(\hat{\pi}_{a(b)}, \pi_{a(b)}) = E(\hat{\pi}_{a(b)} - \pi_{a(b)})^2$,we get
$= MSE(\hat{\pi}_{a(b)}, \pi_{a(b)}) + E(\hat{r} - r)^2(\bar{h}_{ab.} - \bar{h}_{.b.})^2$
by symmetry we get
$= MSE(\hat{\pi}_{a(b)}, \pi_{a(b)}) + \frac{1}{IJK}E[(\hat{r} - r)^2 SS_\pi]$
$E[(\hat{r} - r)^2 SS_\pi] = E\left[\left(\frac{zSS_\epsilon}{SS_\pi} - r\right)^2 SS_\pi\right] = z^2 E\left(\frac{SS_\epsilon^2}{SS_\pi}\right) - 2zrE(SS_\epsilon) + r^2E(SS_\pi)$
$= \sigma_\epsilon^2 r\left\{\frac{z^2J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)} - 2zJ(K-1)(I-1) + J(I-1)\right\}$
and $MSE(\hat{\pi}_{a(b)}, \pi_{a(b)}) = \frac{\sigma_\epsilon^2}{I}\left\{\frac{(1-r)}{rK}[(1-r)(I-1) + 1]\right\}$
then, we get
$$MSE(\hat{\pi}_{1,z;(a)b}, \pi_{a(b)}) = \frac{\sigma_\epsilon^2}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)] + 1\right\} + \frac{\sigma_\epsilon^2 r}{IJK}\left\{\frac{z^2J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)} - 2zJ(K-1)(I-1) + J(I-1)\right\}$$

theorem (2) and (9) can be used to obtain expression for MSEs of untruncated estimators of an arbitrary linear combination of $\theta, \pi_{1(1)}, \dots, \pi_{I(J)}, A_1, \dots, A_J, B_1, \dots, B_K$ and $(AB)_{11}, \dots, (AB)_{JK}$

Corollary 2: For model (1), let z be an arbitrary positive constant and let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimators of $\theta_{abc}(a = 1, \dots, I; b = 1, \dots, J; c = 1, \dots, K)$. Then,

$$MSE\left(\hat{\theta}_{1,z;abc},\theta_{abc}\right)=\frac{\sigma_{\epsilon}^{2}}{K}\left\{\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{r}{K}\left[zJ(K-1)(I-1)\left(\frac{z(IJK-JK-IJ-J+2)}{J(I-3)}-2\right)+J(I-1)\right]+\frac{[r(IK-1)-1]}{rI}\right\} \quad (17)$$

Proof: From theorems (2) and (7 (ii)), we have that

$$MSE\left(\hat{\theta}_{1,z;abc},\theta_{abc}\right)=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\frac{z^{2}J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)}-2zJ(K-1)(I-1)+J(I-1)\right\}+\frac{\sigma_{\epsilon}^{2}[r(IK-1)-1]}{rI^{2}K}$$

$$=\frac{\sigma_{\epsilon}^{2}}{IK}\left\{\frac{(1-r)}{r}[1+(1-r)(I-1)]+\frac{r}{K}\left[zJ(K-1)(I-1)\left(\frac{z(IJK-JK-IJ-J+2)}{J(I-3)}-2\right)+J(I-1)\right]+\frac{r(IK-1)-1}{rI}\right\}$$

Corollary 3: For model (1), let $z^{*}=\frac{J(I-3)}{(IJK-JK-IJ-J+2)}$ be an arbitrary positive constant and let $\hat{\theta}_{1,z;abc}$ be the corresponding type 1 estimators of $\theta_{abc}(a=1,\ldots,I;\ b=1,\ldots,J; c=1,\ldots,K)$. Then,

$$MSE\left(\hat{\pi}_{1,z^{*};(a)b},\pi_{(a)b}\right)=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\frac{z^{2}J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)}-2zJ(K-1)(I-1)+J(I-1)\right\} \quad (18)$$

Proof: from theorem (9), we have that

$$MSE\left(\hat{\pi}_{1,z^{*};(a)b},\pi_{(a)b}\right)=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\frac{z^{2}J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)}\right\}$$

$$MSE\left(\hat{\pi}_{1,z^{*};(a)b},\pi_{(a)b}\right)=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\left(\frac{J(I-3)}{(IJK-JK-IJ-J+2)}\right)^{2}\left[\frac{J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)}-2\left(\frac{J(I-3)}{(IJK-JK-IJ-J+2)}\right)J(K-1)(I-1)+J(I-1)\right]\right\}=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\left[\frac{J(K-1)(I-1)J(I-3)}{(IJK-JK-IJ-J+2)}-2\left(\frac{J(K-1)(I-1)J(I-3)}{(IJK-JK-IJ-J+2)}\right)+J(I-1)\right]\right\}=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\left[-\frac{J(K-1)(I-1)J(I-3)}{(IJK-JK-IJ-J+2)}+J(I-1)\right]\right\}$$

$$=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\left[-\frac{J(K-1)(I-1)J(I-3)+J(I-1)(IJK-JK-IJ-J+2)}{(IJK-JK-IJ-J+2)}\right]\right\}$$

$$=\frac{\sigma_{\epsilon}^{2}}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\}+\frac{\sigma_{\epsilon}^{2}r}{IJK}\left\{\left[\frac{2J[(I-1)(JK-2J+1)-JK]}{(IJK-JK-IJ-J+2)}\right]\right\}$$

Unconditional Properties of Type 2 Estimators

Theorem 10: For model (1), let $\hat{H}_2$ be an arbitrary type 2 estimator of H Then, if $I>2$, $\hat{H}_2$estimates H unbiasedly.

Proof: Let $\hat{\pi}_{2;1(1)}$ and $\hat{r}_2$ be the corresponding type 2 estimators of $\pi_{a(b)}$ and r, respectively. Let $\hat{r}_1$ be the corresponding type 1 estimators of r ,this mean that $\hat{r}_2 = \min\{\hat{r}_2, 1\}$.

According to corollary 2.1 $\widehat{H}_2$ estimates H unbiased at $n > 2$. But we remained prove that $E\left[\left|\hat{\pi}_{2;a(b)}\right|\right] < \infty$. By using (3.11), note that

$E\left[\left|\hat{\pi}_{2;a(b)}\right|\right] < \infty$ iff $E[\hat{r}_2|\bar{h}_{ab.} - \bar{h}_{.b.}|] < \infty$. We get

$E[\hat{r}_2|\bar{h}_{ab.} - \bar{h}_{.b.}|] \leq E[\hat{r}_1|\bar{h}_{ab.} - \bar{h}_{.b.}|]$.

Theorem 3.: For model (1), let $\hat{r}_\delta$ be an arbitrary type 2 estimator r. Take $\hat{r}_\Theta = \min\{\hat{r}_\delta, 1\}$, and let $\hat{\pi}_{\delta;a(b)} = (1 - \hat{r}_\delta)(\bar{h}_{ab.} - \bar{h}_{.b.})$ and $\hat{\pi}_{\gamma;a(b)} = (1 - \hat{r}_\gamma)(\bar{h}_{ab.} - \bar{h}_{.b.}), (a = 1, \dots, I; b = 1, \dots, J)$. Then,

$$E\left[\left(\hat{\pi}_{\gamma;a(b)} - \pi_{a(b)}\right)^2\right] \leq E\left[\left(\hat{\pi}_{\delta;a(b)} - \pi_{a(b)}\right)^2\right] \qquad (19)$$

With strict inequality if $P(\hat{r}_\delta > 1) > 0$ and $E\left[\left(\hat{\pi}_{\gamma;a(b)} - \pi_{a(b)}\right)^2\right] = \infty$.

Proof: If $\left[\left(\hat{\pi}_{\delta;a(b)} - \pi_{a(b)}\right)^2\right] = \infty$, then inequality (14) is hold.

If $E\left[\left(\hat{\pi}_{\delta;a(b)} - \pi_{a(b)}\right)^2\right] < \infty$ then using Theorem (4),
$MSE\left(\hat{\pi}_{\gamma;a(b)} - \pi_{a(b)}\right) = MSE\left(\hat{\pi}_{a(b)}, \pi_{a(b)}\right) + \frac{1}{IJK} E[(\hat{r}_\gamma - r)^2 SS_\pi]$, and
$MSE\left(\hat{\pi}_{\delta;a(b)} - \pi_{a(b)}\right) = MSE\left(\hat{\pi}_{a(b)}, \pi_{a(b)}\right) + \frac{1}{IJK} E[(\hat{r}_\delta - r)^2 SS_\pi]$, we get
$= \frac{1}{IJK}\{E[(\hat{r}_\gamma - r)^2 SS_\pi] - E[(\hat{r}_\delta - r)^2 SS_\pi]\}$
$= E\{(\hat{r}_\gamma^2 - 2r\hat{r}_\gamma + r^2)SS_\pi - (\hat{r}_\delta^2 - 2r\hat{r}_\delta + r^2)SS_\pi\}$
$= E\{(\hat{r}_\gamma^2 - \hat{r}_\delta^2 - 2r(\hat{r}_\gamma + \hat{r}_\delta)SS_\pi\}$
$= E\{(1 - r_\delta^2 - 2(1 + \hat{r}_\delta)SS_\pi | \hat{r}_\delta > 1\}P(\hat{r}_\delta > 1)$
$= E\{(1 - \hat{r}_\delta^2 - 2r(1 + \hat{r}_\delta)SS_\pi | \hat{r}_\delta > 1\}P(\hat{r}_\delta > 1)$
$= E\{(1 - \hat{r}_\delta^2)SS_\pi | \hat{r}_\delta > 1\}P(\hat{r}_\delta > 1) \leq 0$.

Theorem 12: For model (1), let $\hat{r}_2$ be an arbitrary type 2 estimator of r, and let $\hat{\pi}_{a(b)}$ be the corresponding type 2 estimators of $\pi_{a(b)}$. Then,

$$MSE\left(\hat{\pi}_{2;a(b)}, \pi_{a(b)}\right) = \frac{\sigma_\epsilon^2(1-r)}{rIK} + (1-r)^2 \frac{J(I-1)}{JK(I-1)} - 2r^2 z \frac{J(K-1)(I-1)}{JK(I-1)} I_t(v_\epsilon + 1, v_\pi) + r^2 z^2 \frac{J(K-1)(I-1)[J(I-1)(K-1)+1]}{KJ^2(I-1)^2} I_t(v_\epsilon + 2, v_\pi - 1) - (1 - 2r)\frac{J(I-1)}{JK(I-1)} I_t(v_\epsilon, v_\pi - 1) \qquad (20)$$

Proof: using theories (4) and (5), we have $\hat{r}_2 = \hat{r}_{2,z} = \min\{z\frac{SS_\epsilon}{SS_\pi}, 1\}$
$MSE\left(\hat{\pi}_{2;a(b)}, \pi_{a(b)}\right) = MSE\left(\hat{\pi}_{a(b)}, \pi_{a(b)}\right) + \frac{\sigma_\epsilon^2 jk(a-1)}{r} E[(\min\{\frac{rx}{1-x}, 1\} -$

$r)^2(1-x)]$,

since x is a beta random variable with parameters $v_\epsilon = \frac{J(K-1)(I-1)}{2}$ and $v_\pi = \frac{J(I-1)}{2}$, and put $x = \frac{1}{1+zr}$ we have

$$E\left[\left(\min\left\{\frac{rzx}{1-x},1\right\}-r\right)^2(1-x)\right] = E[r^2z^2x^2(1-x)^{-1} - 2r^2zx + r^2z(1-x) + (1-2r)(1-x)]$$
$$= r^2\int_0^1 x^{v_\epsilon-1}(1-x)^{(v_\pi+1)-1}dx - 2r^2z\int_0^t x^{(v_\epsilon+1)-1}(1-x)^{v_\pi-1}dx + r^2z^2\int_0^t x^{(v_\epsilon+2)-1}(1-x)^{(v_\pi-1)-1}dx + (1-2r)\int_t^1 x^{v_\epsilon-1}(1-x)^{(v_\pi+1)-1}dx$$
$$= (r^2-2r+1)\int_0^1 x^{v_e-1}(1-x)^{(v_\pi+1)-1}dx - 2r^2z\int_0^t x^{(v_\epsilon+1)-1}(1-x)^{v_\pi-1}dx + r^2z^2\int_0^t x^{(v_\epsilon+2)-1}(1-x)^{(v_\epsilon-1)-1}dx - (1-2r)\int_0^t x^{v_\epsilon-1}(1-x)^{(v_\pi+1)-1}dx$$
$$= (1-r)^2\frac{v_\pi}{v_\epsilon+v_\pi} - 2\eta^2z\frac{v_\epsilon}{v_\epsilon+v_\pi}I_t(v_\epsilon+1,v_\pi) + r^2z^2\frac{v_\epsilon(v_\epsilon+1)}{(v_\pi-1)(v_\epsilon+v_\pi)}I_t(v_\epsilon+2,v_\pi-1) - (1-2r)\frac{v_\pi}{v_\epsilon+v_\pi}I_t(v_\epsilon,v_\pi-1)$$
$$\therefore E\left[\left(\min\left\{\frac{rzx}{1-x},1\right\}-r\right)^2(1-x)\right] = (1-r)^2\frac{J(I-1)}{JK(I-1)} - 2r^2z\frac{J(K-1)(I-1)}{JK(I-1)}I_t(v_\epsilon+1,v_\pi) + r^2z^2\frac{J(K-1)(I-1)[J(K-1)(I-1)+1]}{KJ^2(I-1)^2}I_t(v_\epsilon+2,v_\pi-1) - (1-2r)\frac{J(I-1)}{JK(I-1)}I_t(v_\epsilon,v_\pi-1) \quad (21)$$

CONCLUSION
In the unconditional properties of repeated measurements model we have the following conclusions:

a. The type 4 estimators are unbiased, means that the other types are unbiased estimators, and also, if $E(\hat{\pi}_{4;1(1)})$ exists, $\widehat{H}_4$estimators H unbiaseedly.

b. $MSE(\widehat{H}_4,H) = \frac{\sigma_\epsilon^2}{rIJK}\left[\ell_0^2 + \sum_{b=1}^J \ell_b^2(I-1) + \sum_{a=1}^I\sum_{b=1}^J \ell_a^2\ell_b^2[(1-\hat{r}_4)^2(I-1)(1-IJ)] + \sum_{c=1}^J r\ell_c^2(K-1) + \sum_{b=1}^J\sum_{c=1}^K r\ell_j^2\ell_c^2 J(K-1)(J-1)\right]$

c. $MSE(\hat{\pi}_{4;a(b)},\pi_{a(b)}) = MSE(\hat{\pi}_{a(b)},\pi_{a(b)}) + \frac{1}{IJK}E[(\hat{r}_4-r)^2SS_\pi]$
$= \frac{\sigma_\epsilon^2(1-r)}{rIK} + \frac{1}{IK}E[(\hat{r}_4-r)^2SS_\pi]$

d. $TMSE(\widehat{\boldsymbol{\theta}}_4,\boldsymbol{\theta}) = IJK\sum_{a=1}^I\sum_{b=1}^J\sum_{c=1}^K E(\hat{\theta}_{4;abc}-\theta_{4;abc})^2$
$= IJK\ MSE(\hat{\theta}_{4;111},\theta_{4;111})$

e. $MSE(\hat{\theta}_{4;abc},\theta_{4;abc}) = MSE(\hat{\pi}_{4;1(1)},\pi_{1(1)}) + \frac{\sigma_\epsilon^2[r(IK-1)-1]}{rI^2K}$

f. $MSE(\hat{\pi}_{1,z^*;(a)b},\pi_{(a)b}) = \frac{\sigma_\epsilon^2}{K}\frac{(1-r)}{r}\left\{\frac{1}{I}[(1-r)(I-1)]+1\right\} + \frac{\sigma_\epsilon^2 r}{IJK}\left\{\frac{z^2J(K-1)(I-1)(IJK-JK-IJ-J+2)}{J(I-3)} - 2zJ(K-1)(I-1) + J(I-1)\right\}$

g. $MSE(\hat{\pi}_{2;a(b)},\pi_{a(b)}) = \frac{\sigma_\epsilon^2(1-r)}{rIK} + (1-r)^2\frac{J(I-1)}{JK(I-1)} -$

$$2r^2 z \frac{J(K-1)(I-1)}{JK(I-1)} I_t(v_\epsilon + 1, v_\pi) + r^2 z^2 \frac{J(K-1)(I-1)[J(I-1)(K-1)+1]}{KJ^2(I-1)^2} I_t(v_\epsilon + 2, v_\pi - 1) - (1-2r) \frac{J(I-1)}{JK(I-1)} I_t(v_\epsilon, v_\pi - 1)$$

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