# COMPOSITE RUNGE – KUTTA METHOD FOURTH ORDER FOR BASED ON VARIETY OF MEANS BY USING INTUITIONISTIC FUZZY DIFFERENTIAL EQUATIONS

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## Abstract

This paper presents Runge-Kutta (RK) method based on Arithmetic Mean (AM), Geometric Mean (GM), Harmonic Mean (HaM), Heronian Mean (HeM), Root Mean Square (RM), Centroidal Mean (CeM) and Contraharmonic Mean (CoM) to solve the intuitionistic Fuzzy differential equations (IFDEs). The efficiency of these methods has been illustrated through numerical examples of intuitionistic fuzzy differential equations.

### 1. Introduction

The concept of fuzzy set theory has been extended into intuitionistic fuzzy set (IFS) theory by Atanassov [2 – 4]. The fuzzy sets theory was considered to be one of Intuitionistic fuzzy set (IFS) theory. Now-a-days, IFSs are being studied widely used in different fields of biology, engineering, physics as well as among other field of science. The studies on improvement of IFS theory, together with intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, an intuitionistic fuzzy approach to artificial intelligence, and intuitionistic fuzzy generalized nets have been given in [14].

Many authors have contributed the numerical solution of fuzzy differential equations with intuitionistic fuzzy initial value problems as follows: Numerical solution of FDE by Runge-Kutta method with intuitionistic treatment treated by Abbasbandy and Allahviranloo [1]. Murugesan et al. have discussed a comparison of extended Runge Kutta

formulae based on variety of means to solve system of IVPs[16]. Melliani et al. [5-8] have discussed differential and partial differential equations under intuitionistic fuzzy environment. Sneh Lata and Amit Kumar [13] have introduced time dependent intuitionistic fuzzy linear differential equation. The solution of intuitionistic fuzzy ordinary differential equation and introduced second order linear differential equations using the fuzzy boundary value by Sankar Prasad Mondal and Tapan Kumar Roy in [9, 12]. The convergence and stability of fuzzy initial value problems discussed by Kelava [4] and Ma et al. [8].

Sneh Lata and Amit Kumar [13] have introduced time dependent intuitionistic fuzzy linear differential equation and have proposed a method to solve it. Nirmala and Chenthur Pandian have studied Intuitionistic fuzzy differential equation with initial condition by using Euler method [11]. Ettoussi et al. [7] has been discussed by the existence and uniqueness of a solution of the intuitionistic fuzzy differential equation using the method of successive approximation.

This paper presents RK methods based on variety of means such as AM, CeM, CoM, HaM, HeM, RM and GM for solving intuitionistic fuzzy IVPs. The efficiency of these methods has been illustrated by numerical examples. For completeness, the RK methods using variety of means for ODEs has also been provided.

## 2. Preliminaries
### Definition 1
Let a set $X$ be fixed. An IFS $A$ in $X$ is an object having the form $A = \{< x, \mu_A(x), \vartheta_A(x) > 0 : x \in X\}$ where the $\mu_A(x) : X \rightarrow [0,1]$ and $\vartheta_A(x) : X \rightarrow [0,1]$ define the degree of membership and degree of non-membership respectively, the element $x \in X$ to the set $A$ which is a subset of $X$, for every element of $x \in X$, $0 < \mu_A(x), \vartheta_A(x) \leq 1$

### Definition 2
An IFN $A$ is defined as follows
- an intuitionistic fuzzy subset of real line.
- normal. i.e., there is any $x_0 \in R$ such that $\mu_A(x) = 1$ (so $\vartheta_{\tilde{A}^i}(x) = 0$)
- a convex set for the membership function $\mu_A(x)$ i.e.,
  $$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \ \forall x_1, x_2 \in R, \lambda \in [0,1]$$
- a concave set for the non-membership function $\vartheta_{\tilde{A}^i}(x)$, i.e.,
  $$\vartheta_A(\lambda x_1 + (1-\lambda) x_2) \leq \min(\vartheta_A(x_1), \vartheta_A(x_2)) \ \forall x_1, x_2 \in R, \lambda \in [0,1].$$

### Definition 3
The $\alpha$, $\beta$-cut of an IFN $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ is defined as follows:
$$A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X, \mu_A(x) \geq \alpha \ and \ \vartheta_A(x) \leq 1-\alpha\} \ \forall x \in [0,1]$$
The $\alpha$-cut representation of IFN A generates the following pair of intervals and is denoted by $[A]_\alpha = \{[A_L^+(\alpha), A_U^+(\alpha)], [A_L^-(\beta), A_U^-(\beta)]\}$.

### Definition 4
An intuitionistic fuzzy set $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ such that $\mu_A(x)$ and $(1-\vartheta_A)(x) = 1 - \vartheta_A(x), \forall x \in R$ are fuzzy numbers, is called an intuitionistic fuzzy number.

Therefore IFS $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ is a conjecture of two fuzzy numbers, $A^+$ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and $A^-$ with a membership function $\mu_{A^-}(x) = 1 - \vartheta_A(x)$.

**Definition 5**

A Triangular Intuitionistic Fuzzy Number (TIFN) A is an intuitionistic fuzzy set in R with the following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$. given as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases} \qquad \vartheta_A(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'}, & a_1' \leq x \leq a_2 \\ \dfrac{x - a_2}{a_3' - a_2}, & a_2 \leq x \leq a_3' \\ 1, & \text{otherwise} \end{cases}$$

Where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$ and TIFN is denoted by $A = (a_1, a_2, a_3; a_1', a_2, a_3')$

Arithmetic operators over TIFNs can be found in [8].

**Definition 6**

For arbitrary $u, v \in E^n$, the quantity $D(u, v) = \sup - 0 \leq \alpha \leq 1 \, d([u]^\alpha, [v]^\alpha)$ is the distance between u and v, where d is the Hausdroff metric in $E^n$.

**Definition 7**

Let mapping $f : I \to W^n$ for some interval $I$ be an intuitionistic fuzzy function. The $\alpha, \beta - \text{cut}$ of $f$ is given by $[f(t)]_{\alpha, \beta} = \left\{ [\underline{f^+}(t; \alpha), \overline{f^+}(t; \alpha)], [\underline{f^-}(t; \beta), \overline{f^-}(t; \beta)] \right\}$,

where

$\underline{f^+}(t; \alpha) = \min\left\{ f^+(t; \alpha) \mid t \in I, 0 \leq \alpha \leq 1 \right\}, \quad \overline{f^+}(t; \alpha) = \max\left\{ f^+(t; \alpha) \mid t \in I, 0 \leq \alpha \leq 1 \right\},$

$\underline{f^-}(t; \alpha) = \min\left\{ f^-(t; \beta) \mid t \in I, 0 \leq \beta \leq 1 \right\}, \quad \overline{f^-}(t; \alpha) = \max\left\{ f^-(t; \beta) \mid t \in I, 0 \leq \beta \leq 1 \right\}.$

## 3. Intuitionistic Fuzzy Cauchy Problem

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a, b] \\ y(t_0) = y_0 \end{cases}$$

(3.1)

where y is an intuitionistic fuzzy function of the crisp variable $t$, $f(t, y(t))$ is an intuitionistic fuzzy function of the crisp variable $t$ and the intuitionistic fuzzy variable y and $y'$ is the intuitionistic fuzzy derivative. If an initial value $y(t_0) = y_0$ {intuitionistic fuzzy number}, we get an intuitionistic fuzzy Cauchy problem of first order $y'(t) = f(t, y(t)), \; y(t_0) = y_0$

As each intuitionistic fuzzy number is a conjecture two fuzzy numbers, equation (3.1) can be replaced by an equivalent system as follows:

$$y'(t) = \left\{ \underline{y}'^{+}(t;\alpha), \overline{y}'^{+}(t;\alpha)], [\underline{y}'^{-}(t;\alpha), \overline{y}'^{-}(t;\alpha)] \right\} \text{, where}$$

$$\underline{y}'^{+}(t;\alpha) = \underline{f}^{+}(t, y^{+}) = \min\left\{ f^{+}(t.u) \mid u \in [\underline{y}^{+}, \overline{y}^{+}] \right\}$$

$$= F(t, \underline{y}^{+}, \overline{y}^{+}), \quad \underline{y}^{+}(t_0) = \underline{y_0}^{+} \tag{3.2}$$

$$\overline{y}'^{+}(t;\alpha) = \overline{f}^{+}(t, y^{+}) = \max\left\{ f^{+}(t.u) \mid u \in [\underline{y}^{+}, \overline{y}^{+}] \right\}$$

$$= G(t, \underline{y}^{+}, \overline{y}^{+}), \quad \overline{y}^{+}(t_0) = \overline{y_0}^{+} \tag{3.3}$$

$$\underline{y}'^{-}(t;\alpha) = \underline{f}^{-}(t, y^{-}) = \min\left\{ f^{-}(t.u) \mid u \in [\underline{y}^{-}, \overline{y}^{-}] \right\}$$

$$= H(t, \underline{y}^{-}, \overline{y}^{-}), \quad \underline{y}^{-}(t_0) = \underline{y_0}^{-} \tag{3.4}$$

$$\overline{y}'^{-}(t;\alpha) = \overline{f}^{-}(t, y^{-}) = \max\left\{ f^{-}(t.u) \mid u \in [\underline{y}^{-}, \overline{y}^{-}] \right\}$$

$$= I(t, \underline{y}^{-}, \overline{y}^{-}), \quad \overline{y}^{-}(t_0) = \overline{y_0}^{-} \tag{3.5}$$

The system of equations given in (3.2) and (3.3) will have unique solution $[\underline{y}^{+}, \overline{y}^{+}] \in B = \overline{c}[0,1] \times \overline{c}[0,1]$ and the system of equations given in (3.4) and (3.5) will have unique solution $[\underline{y}^{-}, \overline{y}^{-}] \in B = \overline{c}[0,1] \times \overline{c}[0,1]$

Therefore, the system given from equations (3.2) to (3.5) possesses unique solution $y(t) = \left\{ [\underline{y}^{+}(t), \overline{y}^{+}(t)], [\underline{y}^{-}(t), \overline{y}^{-}(t)] \right\} \in B \times B$ which is an intuitionistic fuzzy function.

(i. e) for each t, $y(t;\alpha) = \left\{ [\underline{y}^{+}(t;\alpha), \overline{y}^{+}(t;\alpha)], [\underline{y}^{-}(t;\alpha), \overline{y}^{-}(t;\alpha)] \right\}, \alpha \in [0,1]$ is an intuitionistic fuzzy number.

The parametric form of the system of equations (3.2) to (3.5) is given by

$$\underline{y}'^{+}(t;\alpha) = F(t, \underline{y}^{+}(t;\alpha), \overline{y}^{+}(t;\alpha)), \quad \underline{y}^{+}(t_0;\alpha) = \underline{y_0}^{+}(\alpha)$$

$$\overline{y}'^{+}(t;\alpha) = G(t, \underline{y}^{+}(t;\alpha), \overline{y}^{+}(t;\alpha)), \quad \overline{y}^{+}(t_0;\alpha) = \overline{y_0}^{+}(\alpha)$$

$$\underline{y}'^{-}(t;\alpha) = H(t, \underline{y}^{-}(t;\alpha), \overline{y}^{-}(t;\alpha)), \quad \underline{y}^{-}(t_0;\alpha) = \underline{y_0}^{-}(\alpha)$$

$$\overline{y}'^{-}(t;\alpha) = I(t, \underline{y}^{-}(t;\alpha), \overline{y}^{-}(t;\alpha)), \quad \overline{y}^{-}(t_0;\alpha) = \overline{y_0}^{-}(\alpha)$$

for $\alpha \in [0,1]$.

## 4. Fourth Order Runge-Kutta Methods for Solving Intuitionistic fuzzy Differential Equations

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$y' = f(t, y), \qquad y(t_0) = y_0$$

(4.1)

The construction of Runge-Kutta methods is of the form

$$y_{n+1} = y_n + \int_0^h f(t_n + \tau, y(t_n + \tau)) \, d\tau$$

(4.2)

Then the integral in (4.2) is approximated by a quadrature formula can be written as

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i K_i \tag{4.3a}$$

where $K_i = f(t_n + c_i h, \ y_n + h \sum_{j=1}^{i-1} a_{ij} K_j), \quad i = 1, 2, \cdots, s$ (4.3b)

This will be called as the general s-stage Runge - Kutta method. It can be represented in the following tableau format:

$$\begin{array}{c|c} c_i & a_{ij} \\ \hline & b_i \end{array}$$

For explicit methods we restrict ourselves $a_{ij} = 0$ for $i \le j$. It is worth to mention that $b_i = b_i(0)$ and $a_{ij} = a_{ij}(0)$. The explicit RK method satisfies

$$\sum_{i=1}^s b_i(0) = 1, \quad \sum_{j=1}^s a_{ij}(0) = c_i, \quad i = 1, 2, \cdots, s.$$

The parametric form of the system of intuitionistic fuzzy differential equations is given by

$$\underline{y}'^+(t;\alpha) = F(t, \underline{y}^+(t;\alpha), \overline{y}^+(t;\alpha)), \quad \underline{y}^+(t_0;\alpha) = \underline{y_0}^+(\alpha)$$

$$\overline{y}'^+(t;\alpha) = G(t, \underline{y}^+(t;\alpha), \overline{y}^+(t;\alpha)), \quad \overline{y}^+(t_0;\alpha) = \overline{y_0}^+(\alpha)$$

$$\underline{y}'^-(t;\beta) = H(t, \underline{y}^-(t;\beta), \overline{y}^-(t;\beta)), \quad \underline{y}^-(t_0;\beta) = \underline{y_0}^-(\beta)$$

$$\overline{y}'^-(t;\beta) = I(t, \underline{y}^-(t;\beta), \overline{y}^-(t;\beta)), \quad \overline{y}^-(t_0;\beta) = \overline{y_0}^-(\beta)$$

for $\beta \in [0,1]$.

The unique solution of the system of equation $y(t) = \left\{ [\underline{y}^+(t), \overline{y}^+(t)], [\underline{y}^-(t), \overline{y}^-(t)] \right\} \in B \times B$ which is an intuitionistic fuzzy function of t, $y(t;\alpha,\beta) = \left\{ [\underline{y}^+(t;\alpha), \overline{y}^+(t;\alpha)], [\underline{y}^-(t;\beta), \overline{y}^-(t;\beta)] \right\}, \alpha, \beta \in [0,1]$ is an intuitionistic fuzzy number.

Fourth order RK for intuitionistic fuzzy IVPs

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + h \sum_{i=1}^4 b_i K_i \quad \overline{y}^+(t_{n+1};\alpha) = \overline{y}^+(t_n;\alpha) + h \sum_{i=1}^4 b_i L_i$$

$$\underline{y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + h \sum_{i=1}^4 b_i M_i \quad \overline{y}^-(t_{n+1};\beta) = \overline{y}^-(t_n;\beta) + h \sum_{i=1}^4 b_i N_i$$

`(4.4a)

where

$$K_i = F(t_n + c_i h, \ \underline{y}^+(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j K_j),$$

$$L_i = G(t_n + c_i h, \ \overline{y}^+(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j L_j),$$

$$M_i = H(t_n + c_i h, \ \underline{y}^+(t_n;\beta) + h\sum_{j=1}^{i-1} a_j M_j), \quad \text{(4.4b)}$$

$$N_i = I(t_n + c_i h, \ \overline{y}^+(t_n;\beta) + h\sum_{j=1}^{i-1} a_j N_j), \quad i = 1, 2, ..., 4.$$

Where $a_j$'s and $c_i$'s are constants.

## 3.1 RK4 for Intuitionistic Fuzzy IVP

RK4 Arithmetic mean formula for intuitionistic fuzzy IVP is given by

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{K_i + K_{i+1}}{2} = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(AM)$$

$$\overline{y}^+(t_{n+1},\alpha) = \overline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{L_i + L_{i+1}}{2} = \overline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(AM) \quad \text{(4.5a)}$$

$$\overline{y}^-(t_{n+1};\beta) = \overline{y}_n^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{M_i + M_{i+1}}{2} = \overline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(AM)$$

$$\underline{y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{N_i + N_{i+1}}{2} = \underline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(AM)$$

It is written in the form

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$\overline{y}^+(t_{n+1};\alpha) = \overline{y}^+(t_n;\alpha) + \frac{h}{6}\left[L_1 + 2L_2 + 2L_3 + L_4\right]$$

$$\underline{y}^-(t_{n+1};\alpha) = \underline{y}^-(t_n;\alpha) + \frac{h}{6}\left[M_1 + 2M_2 + 2M_3 + M_4\right]$$

$$\overline{y}^-(t_{n+1};\alpha) = \overline{y}^-(t_n;\alpha) + \frac{h}{6}\left[N_1 + 2N_2 + 2N_3 + N_4\right]$$

RK4 Centroidal mean formula for intuitionistic fuzzy IVP is given by

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{2(K_i^2 + K_i K_{i+1} + K_{i+1}^2)}{3(K_i + K_{i+1})} = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(CeM)$$

$$\overline{y}^+(t_{n+1},\alpha) = \overline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{2(L_i^2 + L_i L_{i+1} + L_{i+1}^2)}{3(L_i + L_{i+1})} = \overline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(CeM) \quad \text{(4.6)}$$

$$\overline{y}^-(t_{n+1};\beta) = \overline{y}_n^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{2(M_i^2 + M_i M_{i+1} + M_{i+1}^2)}{3(M_i + M_{i+1})} = \overline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(CeM)$$

$$\underline{y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{2(N_i^2 + N_i N_{i+1} + N_{i+1}^2)}{3(N_i + N_{i+1})} = \underline{y}^-(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(CeM)$$

RK4 Contra-harmonic mean formula for intuitionistic fuzzy IVP is given by

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{K_i^2 + K_{i+1}^2}{K_i + K_{i+1}} = \underline{y}^+(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(CoM)$$

$$\overline{y}^{+}(t_{n+1},\alpha) = \overline{y}^{+}(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}\frac{L_i^2 + L_{i+1}^2}{L_i + L_{i+1}} = \overline{y}^{+}(t_n;\alpha) + \frac{h}{3}\sum_{i=1}^{3}(CoM) \qquad (4.7)$$

$$\overline{y}^{-}(t_{n+1};\beta) = \overline{y}_n^{-}(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{M_i^2 + M_{i+1}^2}{M_i + M_{i+1}} = \overline{y}^{-}(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(CoM)$$

$$\underline{y}^{-}(t_{n+1};\beta) = \underline{y}^{-}(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}\frac{N_i^2 + N_{i+1}^2}{N_i + N_{i+1}} = \underline{y}^{-}(t_n;\beta) + \frac{h}{3}\sum_{i=1}^{3}(CoM)$$

Where

$$K_i = F(t_n + c_i h, \ \underline{y}^{+}(t_n;\alpha) + h\sum_{j=1}^{i-1}a_{ij}K_j),$$

$$L_i = G(t_n + c_i h, \ \overline{y}^{+}(t_n;\alpha) + h\sum_{j=1}^{i-1}a_{ij}L_j),$$

$$M_i = H(t_n + c_i h, \ \underline{y}^{+}(t_n;\beta) + h\sum_{j=1}^{i-1}a_{ij}M_j),$$

$$N_i = I(t_n + c_i h, \ \overline{y}^{+}(t_n;\beta) + h\sum_{j=1}^{i-1}a_{ij}N_j), \quad i = 1,2,3,4.$$

The values of AM can be replaced by variety of means such as GM, HaM, HeM, RM, CeM and CoM. For convenience, we take $a_1 = a_{21}$, $a_2 = a_{31}$, $a_3 = a_{32}$, $a_4 = a_{41}$, $a_5 = a_{42}$ and $a_6 = a_{43}$. The missing elements in the matrix $A = (a_{ij})$, i,j=1,2,3,4 are defined to be zero. The formulae for HaM, HeM,RM, CeM and CoM in terms of AM and GM and also the values for $a_{ij}$ is for these means can be referred in [22]. The fourth order ERK formulae based on various means for solving fuzzy IVPs are given in Table 4.1, Table 4.2, Table 4.3 and Table 4.4.

**Table 4.1**

| S. No. | Means | $\underline{y}^{+}_{n+1}$ |
|--------|-------|---------------------------|
| 1. | AM | $\underline{y}_n^{+} + \dfrac{h}{6}\left[K_1 + 2(K_2 + K_3) + K_4\right]$ |
| 2. | GM | $\underline{y}_n^{+} + \dfrac{h}{3}\left[\sqrt{|K_1 K_2|} + \sqrt{|K_2 K_3|} + \sqrt{|K_3 K_4|}\right]$ |
| | HaM | $\underline{y}_n^{+} + \dfrac{2h}{3}\left[\dfrac{K_1 K_2}{K_1 + K_2} + \dfrac{K_2 K_3}{K_2 + K_3} + \dfrac{K_3 K_4}{K_3 + K_4}\right]$ |
| 4. | HeM | $\underline{y}_n^{+} + \dfrac{h}{9}\left[K_1 + 2(K_2 + K_3) + K_4 + \sqrt{|K_1 K_2|} + \sqrt{|K_2 K_3|} + \sqrt{|K_3 K_4|}\right]$ |
| 5. | RM | $\underline{y}_n^{+} + \dfrac{h}{3}\left[\sqrt{\dfrac{K_1^2 + K_2^2}{2}} + \sqrt{\dfrac{K_2^2 + K_3^2}{2}} + \sqrt{\dfrac{K_3^2 + K_4^2}{2}}\right]$ |
| 6. | CeM | $\underline{y}_n^{+} + \dfrac{2}{9}h\left[\dfrac{K_1^2 + K_1 K_2 + k_2^2}{K_1 + K_2} + \dfrac{K_2^2 + K_2 K_3 + K_3^2}{K_2 + K_3} + \dfrac{K_3^2 + K_3 K_4 + K_4^2}{K_3 + K_4}\right]$ |

| S. No. | Means | |
|---|---|---|
| 7. | CoM | $\underline{y}_n^+ + \dfrac{h}{3}\left[\dfrac{K_1^2 + K_2^2}{K_1 + K_2} + \dfrac{K_2^2 + K_3^2}{K_2 + K_3} + \dfrac{K_3^2 + K_4^2}{K_3 + K_4}\right]$ |

**Table 4.2**

| S. No. | Means | $\overline{y}_{n+1}^+$ |
|---|---|---|
| 1. | AM | $\overline{y}_n^+ + \dfrac{h}{6}\left[L_1 + 2(L_2 + L_3) + L_4\right]$ |
| 2. | GM | $\overline{y}_n^+ + \dfrac{h}{3}\left[\sqrt{|L_1 L_2|} + \sqrt{|L_2 L_3|} + \sqrt{|L_3 L_4|}\right]$ |
| 3. | HaM | $\overline{y}_n^+ + \dfrac{2h}{3}\left[\dfrac{L_1 L_2}{L_1 + L_2} + \dfrac{L_2 L_3}{L_2 + L_3} + \dfrac{L_3 L_4}{L_3 + L_4}\right]$ |
| 4. | HeM | $\overline{y}_n^+ + \dfrac{h}{9}\left[L_1 + 2(L_2 + L_3) + L_4 + \sqrt{|L_1 L_2|} + \sqrt{|L_2 L_3|} + \sqrt{|L_3 L_4|}\right]$ |
| 5. | RM | $\overline{y}_n^+ + \dfrac{h}{3}\left[\sqrt{\dfrac{L_1^2 + L_2^2}{2}} + \sqrt{\dfrac{L_2^2 + L_3^2}{2}} + \sqrt{\dfrac{L_3^2 + L_4^2}{2}}\right]$ |
| 6. | CeM | $\overline{y}_n^+ + \dfrac{2}{9}h\left[\dfrac{L_1^2 + L_1 L_2 + L_2^2}{L_1 + L_2} + \dfrac{L_2^2 + L_2 L_3 + k_3^2}{L_2 + L_3} + \dfrac{L_3^2 + L_3 L_4 + L_4^2}{L_3 + l_4}\right]$ |
| 7. | CoM | $\underline{y}_n^+ + \dfrac{h}{3}\left[\dfrac{L_1^2 + L_2^2}{L_1 + L_2} + \dfrac{L_2^2 + L_3^2}{L_2 + L_3} + \dfrac{L_3^2 + L_4^2}{L_3 + L_4}\right]$ |

**Table 4. 3**

| S. No. | Means | $\underline{y}_{n+1}^-$ |
|---|---|---|
| 1. | AM | $\underline{y}_n^- + \dfrac{h}{6}\left[M_1 + 2(M_2 + M_3) + M_4\right]$ |
| 2. | GM | $\underline{y}_n^- + \dfrac{h}{3}\left[\sqrt{|M_1 M_2|} + \sqrt{|M_2 M_3|} + \sqrt{|M_3 M_4|}\right]$ |
| 3. | HaM | $\underline{y}_n^- + \dfrac{2h}{3}\left[\dfrac{M_1 M_2}{M_1 + M_2} + \dfrac{M_2 M_3}{M_2 + M_3} + \dfrac{M_3 M_4}{M_3 + M_4}\right]$ |
| 4. | HeM | $\underline{y}_n^- + \dfrac{h}{9}\left[M_1 + 2(M_2 + M_3) + M_4 + \sqrt{|M_1 M_2|} + \sqrt{|M_2 M_3|} + \sqrt{|M_3 M_4|}\right]$ |
| 5. | RM | $\underline{y}_n^- + \dfrac{h}{3}\left[\sqrt{\dfrac{M_1^2 + M_2^2}{2}} + \sqrt{\dfrac{M_2^2 + M_3^2}{2}} + \sqrt{\dfrac{M_3^2 + M_4^2}{2}}\right]$ |

| 6. | CeM | $\underline{y}_n^- + \dfrac{2}{9}h\left[\dfrac{M_1^2 + M_1 M_2 + M_2^2}{M_1 + M_2} + \dfrac{M_2^2 + M_2 M_3 + M_3^2}{M_2 + M_3} + \dfrac{M_3^2 + M_3 M_4 + M_4^2}{M_3 + M_4} - \right]$ |
|---|---|---|
| 7. | CoM | $\underline{y}_n^- + \dfrac{h}{3}\left[\dfrac{M_1^2 + M_2^2}{M_1 + M_2} + \dfrac{M_2^2 + M_3^2}{M_2 + M_3} + \dfrac{M_3^2 + M_4^2}{M_3 + M_4}\right]$ |

**Table 4.4**

| S. No. | Means | $\overline{y}_{n+1}^{-+}$ |
|---|---|---|
| 1. | AM | $\overline{y}_n^- + \dfrac{h}{6}\left[N_1 + 2(N_2 + N_3) + N_4\right]$ |
| 2. | GM | $\overline{y}_n^- + \dfrac{h}{3}\left[\sqrt{|N_1 N_2|} + \sqrt{|N_2 N_3|} + \sqrt{|N_3 N_4|}\right]$ |
| 3. | HaM | $\overline{y}_n^- + \dfrac{2h}{3}\left[\dfrac{N_1 N_2}{N_1 + N_2} + \dfrac{N_2 N_3}{N_2 + N_3} + \dfrac{N_3 N_4}{N_3 + N_4}\right]$ |
| 4. | HeM | $\overline{y}_n^- + \dfrac{h}{9}\left[N_1 + 2(N_2 + N_3) + N_4 + \sqrt{|N_1 N_2|} + \sqrt{|N_2 N_3|} + \sqrt{|N_3 N_4|}\right]$ |
| 5. | RM | $\overline{y}_n^- + \dfrac{h}{3}\left[\sqrt{\dfrac{N_1^2 + N_2^2}{2}} + \sqrt{\dfrac{N_2^2 + N_3^2}{2}} + \sqrt{\dfrac{N_3^2 + N_4^2}{2}}\right]$ |
| 6. | CeM | $\overline{y}_n^- + \dfrac{2}{9}h\left[\dfrac{N_1^2 + N_1 N_2 + N_2^2}{N_1 + N_2} + \dfrac{N_2^2 + N_2 N_3 + N_3^2}{N_2 + N_3} + \dfrac{N_3^2 + N_3 N_4 + N_4^2}{N_3 + N_4}\right]$ |
| 7. | CoM | $\underline{y}_n^- + \dfrac{h}{3}\left[\dfrac{N_1^2 + N_2^2}{N_1 + N_2} + \dfrac{N_2^2 + N_3^2}{N_2 + N_3} + \dfrac{N_3^2 + N_4^2}{N_3 + N_4}\right]$ |

## 5. Convergence of Intuitionistic Fuzzy Runge-Kutta Methods

The solution is obtained by grid points at

$$a = t_0 \le t_1 \le \cdots \le t_N = b \text{ and } h = \frac{b-a}{N} = t_{n+1} - t_n \qquad (5.1)$$

We define

$$F[t_n, y(t_n; a)] = \sum_{i=1}^{s} b_i K_i(t_n, y(t_n; a)) \quad G[t_n, y(t_n; a)] = \sum_{i=1}^{s} b_i L_i(, y(t_n; a))$$

$$H[t_n, y(t_n; \beta)] = \sum_{i=1}^{s} b_i M_i(t_n, y(t_n; \beta)) \quad I[t_n, y(t_n; \beta)] = \sum_{i=1}^{s} b_i N_i(t_n, y(t_n; \beta)) \qquad (5.2)$$

The exact and approximate solutions at $t_n$, $0 \le n \le N$ are denoted respectively by

$$[Y(t_n)]_{\alpha,\beta} = [\underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha), \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)] \text{ and}$$

$$[y(t_n)]_{\alpha,\beta} = [\underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha), \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)].$$

We have

$$\underline{Y}^+(t_{n+1};\alpha) \approx \underline{Y}^+(t_n;\alpha) + h F[t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)], \quad \overline{Y}^+(t_{n+1};\alpha) \approx \overline{Y}^+(t_n;\alpha) + h G[t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)],$$

$$\underline{Y}^-(t_{n+1};\beta) \approx \underline{Y}^-(t_n;\beta) + h H[t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)], \quad \overline{Y}^-(t_{n+1};\beta) \approx \overline{Y}^-(t_n;\beta) + h I[t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)].$$

and

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + h F[t_n, \underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha)], \quad \overline{Y}^+(t_{n+1};\alpha) = \overline{y}^+(t_n;\alpha) + h G[t_n, \underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha)],$$

$$\underline{Y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + h H[t_n, \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)], \quad \overline{Y}^-(t_{n+1};\beta) = \overline{y}^-(t_n;\beta) + h I[t_n, \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)].$$

We need the following lemmas to show the convergence of these approximates, that is,

$\underline{y}^+(t_n;\alpha)$, $\overline{y}^+(t_n;\alpha)$, $\underline{y}^-(t_n;\alpha)$, and $\overline{y}^-(t_n;\alpha)$ converges

to $\underline{Y}^+(t_n;\alpha)$, $\overline{Y}^+(t_n;\alpha)$, $\underline{Y}^-(t_n;\alpha)$, and $\overline{Y}^-(t_n;\alpha)$ respectively whenever $h \to 0$.

**Lemma 5.1**

Let the sequence of numbers $\{W_n^+\}_{n=0}^N, \{W_n^-\}_{n=0}^N$ satisfy

$$|W_{n+1}| \le A|W_n| + B \quad 0 \le n \le N-1$$

for some given positive constants A and B. Then

$$|W_n| \le A^n |W_0| + B\frac{A^n - 1}{A-1}.$$

**Lemma 5.2**

Let the sequence of numbers $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n||V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A.\max\{|W_n||V_n|\} + B,$$

for some given positive constants A and B and denote

$$U_n = |W_n| + |V_n|, \quad 0 \le n \le N.$$

Then

$$U_n \le \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A}-1}, \quad 0 \le n \le N \text{ where } \overline{A} = 1+2A \text{ and } \overline{B} = 2B.$$

Let F (t, u, v) and G (t, u, v) are obtained by substituting $[y(t)]_r = [u,v]$ in (5.2).

The domain where F and G are defined is therefore

$$K = \{(t,u,v)/0 \le t \le T, \quad -\infty < v < \infty, \ -\infty < u \le v\}.$$

**Theorem 5.1**

Let F (t, u, v), G (t, u, v), H (t, u', v') and I (t, u', v') belonging to $c^P(K)$ and let the partial derivatives of F, G, H and I be bounded over K. Then for arbitrary fixed $\alpha, \beta, 0 \le \alpha, \beta \le 1$, the approximate solution of (5.1), $\left[\underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha), \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)\right]$

converge to the exact solution $\left[\underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha), \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)\right]$.

**Proof:** By using Taylor's theorem

$$\underline{Y}^+(t_{n+1};\alpha) = \underline{Y}^+(t_n;\alpha) + hF(\mathrm{t}_n,\underline{Y}^+(t_n;\alpha),\overline{Y}^+(t_n;\alpha)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{+\,(p+1)}(\xi_{n,1})$$

$$\overline{Y}^+(t_{n+1};\alpha) = \overline{Y}^+(t_n;\alpha) + hG(\mathrm{t}_n,\underline{Y}^+(t_n;\alpha),\overline{Y}^+(t_n;\alpha)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{+\,(p+1)}(\xi_{n,2})$$

$$\underline{Y}^-(t_{n+1};\beta) = \underline{Y}^-(t_n;\beta) + hH(\mathrm{t}_n,\underline{Y}^-(t_n;\beta),\overline{Y}^-(t_n;\beta)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{-\,(p+1)}(\xi_{n,3})$$

$$\overline{Y}^-(t_{n+1};\beta) = \overline{Y}^-(t_n;\beta) + hI(\mathrm{t}_n,\underline{Y}^-(t_n;\beta),\overline{Y}^-(t_n;\beta)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{-\,(p+1)}(\xi_{n,4})$$

where $\xi_{n,1},\xi_{n,2},\xi_{n,3},\xi_{n,4} \in (\mathrm{t}_n,t_{n+1})$

Now if we denote

$$W_n^+ = \underline{Y}^+(t_n,\alpha) - \underline{y}^+(t_n,\alpha) , \quad V_n^+ = \overline{Y}^+(t_n,\alpha) - \overline{y}^+(t_n,\alpha)$$

$$W_n^- = \underline{Y}^-(t_n,\beta) - \underline{y}^-(t_n,\beta) \text{ and } V_n^- = \overline{Y}^-(t_n,\beta) - \overline{y}^-(t_n,\beta)$$

then the above two expressions converted to
Hence we can write

$$\left|W_{n+1}^+\right| \le \left|W_n^+\right| + 2Lh\max\left(\left|W_n^+\right|,\left|V_n^+\right|\right) + \frac{h^{p+1}}{(p+1)!}\mathbf{M}_1$$

$$\left|V_{n+1}^+\right| \le \left|V_n^+\right| + 2Lh\max\left(\left|W_n^+\right|,\left|V_n^+\right|\right) + \frac{h^{p+1}}{(p+1)!}\mathbf{M}_1$$

$$\left|W_{n+1}^-\right| \le \left|W_n^-\right| + 2Lh\max\left(\left|W_n^-\right|,\left|V_n^-\right|\right) + \frac{h^{p+1}}{(p+1)!}\mathbf{M}_2$$

$$\left|V_{n+1}^-\right| \le \left|V_n^-\right| + 2Lh\max\left(\left|W_n^-\right|,\left|V_n^-\right|\right) + \frac{h^{p+1}}{(p+1)!}\mathbf{M}_2$$

where

$$M_1 = \max\left\{\max\left|\underline{Y}^{+\,(p+1)}(t,\alpha)\right|,\max\left|\overline{Y}^{+\,(p+1)}(t,\alpha)\right|\right\}$$

$$M_2 = \max\left\{\max\left|\underline{Y}^{-\,(p+1)}(t,\beta)\right|,\max\left|\overline{Y}^{-\,(p+1)}(t,\beta)\right|\right\} \quad for\,t \in [0,\mathrm{T}],$$

and $L > 0$ is a bound from the partial derivative of F and G.
Therefore, we can write,

$$\left|W_n^+\right| \le (1+4\,\mathrm{Lh})^n\left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh},$$

$$\left|V_n^+\right| \le (1+4\,\mathrm{Lh})^n\left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}$$

$$\left|W_n^-\right| \le (1+4\,\mathrm{Lh})^n\left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh},$$

$$\left|V_n^-\right| \le (1+4\,\mathrm{Lh})^n\left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\,\mathrm{Lh})^n-1}{4Lh}$$

where $\left|U_0^+\right| = \left|W_0^+\right| + \left|V_0^+\right|$ and $\left|U_0^-\right| = \left|W_0^-\right| + \left|V_0^-\right|$
In particular

$$\left|W_n^+\right| \le (1+4\,\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh},$$

$$\left|V_n^+\right| \le (1+4\,\mathrm{Lh})^N \left|U_0^+\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}$$

$$\left|W_n^-\right| \le (1+4\,\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh},$$

$$\left|V_n^-\right| \le (1+4\,\mathrm{Lh})^N \left|U_0^-\right| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\,\mathrm{Lh})^{\frac{T}{h}}-1}{4Lh}$$

where

$$M = \max\left\{M_1, M_2\right\} \ for\, t \in [0, \mathrm{T}],$$

since $W_0^+ = V_0^+ = W_0^- = V_0^- = 0,$ we have

$$\left|W_n^+\right| \le M\,\frac{(e^{4LT}-1)}{2L(p+1)!}h^p, \ \ \left|V_n^+\right| \le M\,\frac{(e^{4LT}-1)}{2L(p+1)!}h^p,$$

$$\left|W_n^-\right| \le M\,\frac{(e^{4LT}-1)}{2L(p+1)!}h^p, \ \ \left|V_n^-\right| \le M\,\frac{(e^{4LT}-1)}{2L(p+1)!}h^p$$

Thus if $h \to 0$ we get $W_n^+ \to 0, \ V_n^+ \to 0, W_n^- \to 0$ and $V_n^- \to 0$ which completes the proof.

## Numerical Examples

**Example 4.1.** *Consider the FODE with* $y'(t) = y$ *with* $y(0) = (3,5,7;1.5,5,8)$

**Solution:** The solution is given by

$$\underline{y}^+(t,\alpha) = (3+2\alpha)e^t \quad \overline{y}^+(t,\alpha) = (7-2\alpha)e^t$$

$$\underline{y}^-(t,\beta) = (5-3.5\beta)e^t \quad \overline{y}^-(t,\beta) = (5+3\beta)e^t$$

The Error results for example 1 at t =1 and $(\alpha, \beta) = 0.8$ are shown in the Table 1. The error graphs are shown Figure 1.

**Table 1**

| Means | Absolute Error for IFRK4 at t=1 | | | |
|---|---|---|---|---|
| | $\underline{y}^+(t;\alpha)$ | $\overline{y}^+(t;\alpha)$ | $\underline{y}^-(t;\beta)$ | $\overline{y}^-(t;\beta)$ |
| AM | 1.989e-08 | 1.584e-08 | 2.063e-08 | 1.695e-08 |
| HeM | 1.851e-08 | 2.173e-08 | 1.730e-08 | 2.253e-08 |
| CeM | 8.766e-08 | 1.029e-07 | 8.194e-08 | 1.067e-07 |
| GM | 9.194e-08 | 1.079e-07 | 8.594e-08 | 1.119e-07 |
| RM | 1.226e-07 | 1.439e-07 | 1.146e-07 | 1.492e-07 |

| HaM | 2.154e-07 | 2.528e-07 | 2.013e-07 | 2.622e-07 |
| CoM | 2.619e-07 | 3.074e-07 | 2.448e-07 | 3.188e-07 |

**Fig. 1**
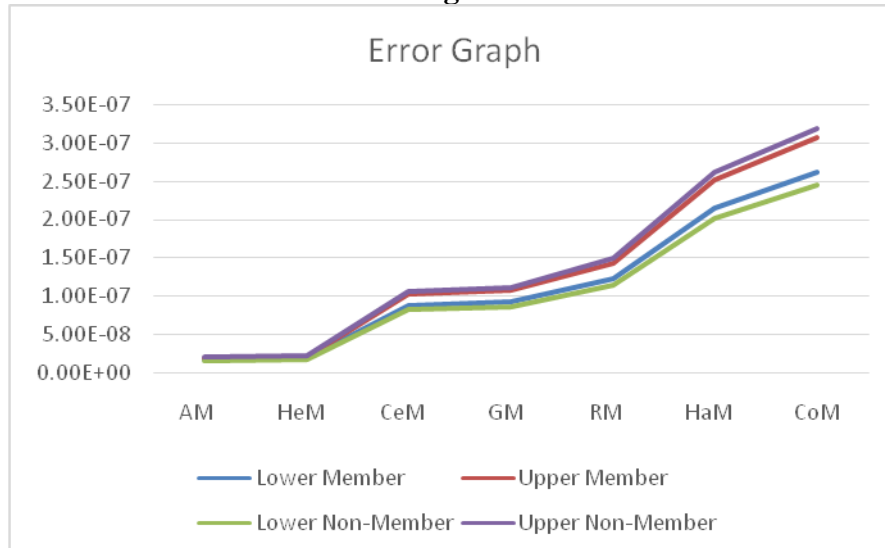


### Example 2 Oil production Model:

In the world rate of increase oil production y in million metric tons per year was assumed to be proportional to y itself. Then what is the amount of oil after five years initially (1061, 1091, 1131; 1066, 1091, 1111) million metric ton oil (the constant of proportionality is 0.084 ).

**Solution:** $\dfrac{dy}{dt} = ky$ with $\quad$ k=0.084 $\quad$ and

$y_0 = (1061,\ 1091,\ 1131;\ 1066,\ 1091,\ 1111)$ million.

The exact solution is given by

$$\underline{y}^+(t;\alpha) = (1061 + 30\alpha)e^{kt} \qquad \overline{y}^+(t;\alpha) = (1131 - 40\alpha)e^{kt}$$

$$\underline{y}^-(t;\beta) = (1091 - 25\alpha)e^{kt} \qquad \overline{y}^-(t;\beta) = (1091 + 20\alpha)e^{kt}$$

The Error results for example 2 at t =1 and $(\alpha, \beta) = 1$ are shown in the Table 1. The error graph are shown Figure 2.

**Table 2**

| Means | Absolute Error for IFRK4 at t=1 | | | |
|---|---|---|---|---|
| | $\underline{y}^+(t;\alpha)$ | $\overline{y}^+(t;\alpha)$ | $\underline{y}^-(t;\beta)$ | $\overline{y}^-(t;\beta)$ |
| AM | 1.179e-09 | 1.192e-09 | 1.180e-09 | 1.188e-09 |
| HeM | 1.179e-09 | 1.192e-09 | 1.180e-09 | 1.188e-09 |
| CeM | 1.179e-09 | 1.192e-09 | 1.180e-09 | 1.188e-09 |
| GM | 1.179e-09 | 1.191e-09 | 1.179e-09 | 1.188e-09 |

| | | | | |
|---|---|---|---|---|
| RM | 1.179e-09 | 1.192e-09 | 1.180e-09 | 1.189e-09 |
| HaM | 1.178e-09 | 1.191e-09 | 1.178e-09 | 1.188e-09 |
| CoM | 1.181e-09 | 1.193e-09 | 1.181e-09 | 1.190e-09 |

**Fig. 2**
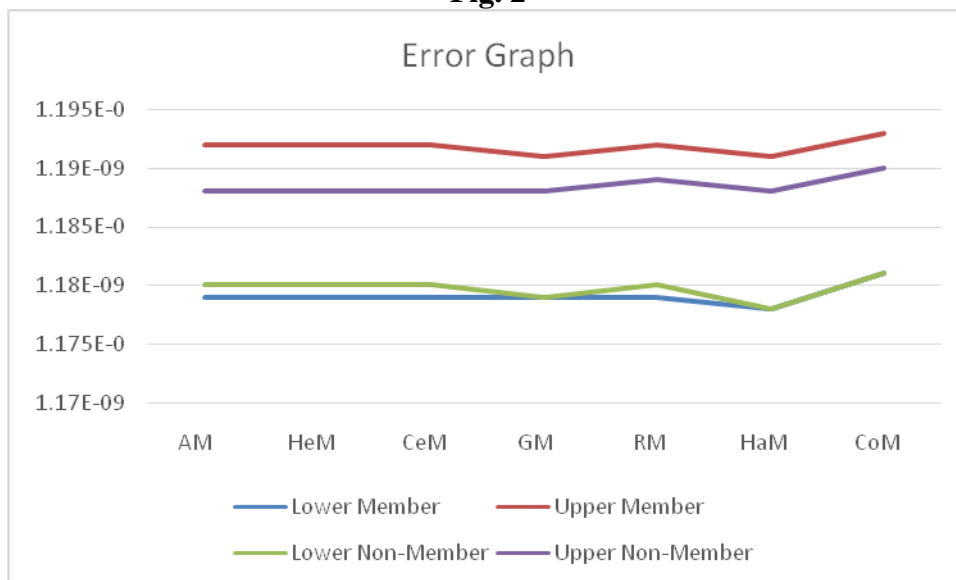


By analyzing the absolute errors from Tables 1 – 2, the RK methods based on variety of means could be sorted under different levels subject to minimum error. The pictorial representation is given in Fig. 3.

**Fig. 3**



**Conclusion**

In this paper, the fourth order Runge Kutta methods based on AM, HeM, CeM, RM, GM, HaM and CoM have been presented to solve the intuitionistic fuzzy IVPs. The efficiency of the method has been illustrated through the numerical examples of Triangular type. The numerical results obtained are agree well with the corresponding exact solutions. This shows that the fourth order RK method based on the above variety of means are much applicable to solve intuitionistic fuzzy IVPs.

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