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# CONSTRUCTION OF UNIT REGULAR MONOIDS

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**ABSTRACT**

Let $G$ be the group of units of a monoid $S$. A unit regular semigroup $S$ should satisfy the condition that corresponding to each $x \in S$ there should exist an element $u \in G$ such that $x = xux$ . Consider $S_0 = S \backslash G$ as the set of non units of $S$. Then evidently $S_0$ is a regular sub semigroup of $S$. Then $S = S_0 \cup G$ $and$ $S_0 \cap G = \emptyset$. Conversely starting from a regular semigroup $S$o and a group $G$ we have constructed a unit regular semigroup with $S$ *with* $G$ as group of units of $S$ and $S_0$as semigroup of non-units of $S$. .

## 1. PRELIMINARIES

For this construction we have introduced the notion of translational hulls. A right translation of a semigroup $S$ is a transformation ρ satisfying the condition that $x(y\rho) = (xy)\rho$ for all $x$, $y$ in $S$. A left translation of a semigroup $S$ is a transformation λ satisfying the condition that $(x\lambda)y = (xy)\lambda$ for all $x$, $y$ in $S$. If $x(y\lambda) = (x\rho)y$ for all $x$, $y$ in S, then we say that a right translation ρ and a left translation λ are linked. Corresponding to each element of the semigroup $S$ we can introduce a transformation $\rho_a(\lambda_a)$ of $S$ given by $x\rho_a = xa$ $[x\lambda_a = ax]$ for all $x$ in $S$. If we consider $T(S)$ as the full transformation semigroup on the set $S$ then evidently these transformations are elements of $T(S)$. For any element $a \in S$ the inner translations $\rho_a$ and $\lambda_a$ are linked. Also the set of left (right) translations can be seen to be a sub semigroup of $T(S)$.

The translational hull $\Omega(S)$ of a semigroup $S$ is defined to be the set of all ordered pairs$(\lambda, \rho)$ of linked left and right translations $\lambda$ and $\rho$ of $S$

We define the translational hull $\Omega(S)$ of a semigroup $S$ to be the set of all pairs $(\lambda, \rho)$ of linked right and left translations $\rho$ and $\lambda$ of $S$. If $(\lambda_1, \rho_1)$, $(\lambda_2, \rho_2) \in \Omega(S)$, then so is $(\lambda_2\lambda_1, \rho_1\rho_2)$ . In $\Omega(S)$ we may define a binary operation by,

$(\lambda_1, \rho_1)$ $(\lambda_2, \rho_2)$ = $(\lambda_2\lambda_1, \rho_1\rho_2)$

Associative property is true and so $\Omega(S)$ is a semigroup. Let $\Omega$o (S) = $\{(\lambda_a, \rho_a); a \in S\}$. Then

$\Omega_0(S) \sqsubseteq \Omega(S)$ since $\rho_a$ and $\lambda_a$ are linked. For any *a* and *b* in *S*, we have

$(\lambda_a, \rho_a)$ $(\lambda_b, \rho_b)$= $(\lambda_b\lambda_a, \rho_a\rho_b)$= $(\lambda_{ab}, \rho_{ab})$.

Starting from a regular semigroup *S*o and a group *G*, the method of constructing a unit regular semigroup is illustrated in the following theorem. With respect to the translational hull *S*o we give the conditions required for this construction .

**2. A CONSTRUCTION**

Given a regular semigroup and a group, we give the conditions required for the construction of unit regular semigroups .

**THEOREM 2.1.** Let *G* be a group and $S_0$ a regular semigroup. Let the translational hull of the semigroup $S_0$be $\Omega(S_0)$ and consider $\Psi$ to be a homomorphism from *G* to $\Omega(S_0)$ defined by $\psi(u) = (\psi_1(u), \psi_2(u))$. Suppose that the following conditions are satisfied by $\Psi$

$\psi_1(1)$ and $\psi_2(1)$ will act as identity permutations on $S_0$.

(i) For every $x \in S_0$, there exists some $u \in G$ such that $(x)\psi_1(u) \in E(S_0)$ (or $(x)\psi_2(u) \in E(S_0)$ , considering $E(S_0)$ as the set of idempotents of the regular semigroup $S_0$

(ii) $\psi_1(u_1)\psi_2(u_2) = \psi_2(u_2)\psi_1(u_1)$ for any two elements $u_{1,}u_2 \in G$.

Define *S* to be the disjoint union of $S_0$ and *G*. That is $S = S_0 \cup G$. The binary operation on *S* can be defined as follows. $ux = (x)\psi_1(u)$ and $xu = (x)\psi_2(u)$ for $x \in S_0$ and $u \in G$. If *u* and *x* are elements of *G* or if they are elements of $S_0$, then the product will be same as that in *G* or $S_0$. Then $S = S_0 \cup G$ will be a unit regular semigroup with semigroup of non units as $S_0$ and group of units as *G*.

**Proof :** We will first show that the associative property holds in *S*. That is we have to prove that

$(i) u(xy) = (ux)y$ for $u \in G$ and $x, y \in S_0$

$(ii)(xy)u = x(yu)$ for $u \in G$ and $x, y \in S_0$

$(iii)x(uy) = (xu)y$ for $u \in G$ and $x, y \in S_0$

$(iv)(u_1u_2)x = u_1(u_2x)$ for $u_1, u_2 \in G$ and $x \in S_0$

(*v*)$(x(u_1u_2) = (xu_1)u_2$ for $u_1, u_2 \in G$ and $x \in S_0$

$(vi)(u_1x)u_2 = u_1(xu_2)$, for $u_1, u_2 \in G$ and $x \in S_0$

Now since $\psi_1(u)$is a left translation,$\big(x\psi_1(u)\big)y = (xy)\Psi1(\text{u})$ . So $u(xy) = (ux)y$ for $u \in G$ and $x, y\in S_0$. Now since $\psi_2(u)$ is a right translation $x(y\psi_2(u))$ = $(xy)\psi_2(u)$ .Therefore $(xy)u = x(yu)$ for $u \in G$ and $x, y \in S_0$. Because $\Psi$ is a homomorphism we get $\psi(u_1u_2) = \big(\psi_1(u_1u_2), \psi_2(u_1u_2)\big)$ . Also we have

$$\psi(u_1)\psi(u_2) \quad = \big(\psi_1(u_1), \psi_2(u_1)\big)\,\big(\psi_1(u_2), \psi_2(u_2)\big)$$

$$= \big(\psi_1(u_2)\psi_1(u_1), \psi_2(u_1)\psi_2(u_2)\big)$$

Hence $\psi_1(u_1u_2) = \psi_1(u_2)\psi_1(u_1)$ and $\psi_2(u_1u_2) = \psi_2(u_1)\psi_2(u_2)$. So $(x)\psi_1(u_1u_2) = (x)[\psi_1(u_2)\psi_1(u_1)]$.

Therefore $(u_1u_2)x = [(x)\psi_1(u_2)]\psi_1(u_1) = u_1(u_2x)$.

In a similar way since $\psi_2(u_1u_2) = \psi_2(u_1)\psi_2(u_2)$we get that $x(u_1u_2) = (xu_1)u_2$ for $u_1, u_2 \in G$ ,$x \in S_0$. Because $\psi_1(u)$ and $\psi_2(u)$ are linked, we get $x\big(y\psi_1(u)\big) = (x\psi_2(u))y$ .So $x(uy) = (xu)y$ for $u \in$ and $x, y \in S_0$. By condition (iii) we have $(x)[\psi_1(u_1)\psi_2(u_2)]$ =$(x)[\psi_2(u_2)\psi_1(u_1)]$ for $u_1, u_2 \in G$ and $x \in S_0$. Therefore $[(x)\psi_1(u_1)]\psi_2(u_2)$ = $[(x)\psi_2(u_2)]\psi_1(u_1)$ .Hence $(u_1x)u_2 = u_1(xu_2)$. So $S$ is a semigroup. Now since $(x)\psi_1(1) = x$ and $(x)\psi_2(1) = x$ for any $x \in S_0$we get that $x1 = x$ and $1x = x$ for every $x \in S_0$. Therefore the identity element of $G$ will be same as the identity element of $S$. Hence $S$ is a monoid.

Now we show that $S$ is unit regular .For $x \in S_0$, let $(x)\psi_1(u) \in E(S_0)$ for some element $u \in G$.. Hence , $ux \in E(S_0)$ So $(ux)(ux) = ux$. So $u(xux) = ux$. Therefore $xux = x$. Hence $S$ will be a unit regular semigroup with group of units as $G$. . □

Next we will show that if $S$ is any unit regular semigroup with $G$ as group of units and $S_0$as semigroup of non-units then all the requirements of the above theorem are satisfied.

**THEOREM 2.2**. Consider $S$ to be a unit regular semigroup. Then there exists a subgroup $G$ of $S$ and a regular sub semigroup $S_0$ of $S$ such that the mapping $\Psi$ from $G$ to $\Omega(S_0)$ given by $\psi(u) = \big(\psi_1(u), \psi_2(u)\big)$ is a homomorphism such that

(i) $\psi_1(1)$and $\psi_2(1)$ act as the identity permutations on $S_0$

(ii) $(x)\psi_1(u)$ [ or $(x)$ $\psi_2(u)$) ]$\in E(S_0)$

(ii) $\psi_1(u_1)\,\psi_2(u_2) = \psi_2(u_2)\,\psi_1(u_1)$ for any $u_1, u_2 \in G$.

**Proof** : Consider $G$ to be the group of units of $S$ and $S_0$ to be the set of all non units of $S$.If $u \in G$ and $x \in S_0$, then $ux$ and $xu \in S_0$.So we can define two functions such as $\psi_1(u)$ and $\psi_2(u)$ defined by

$x\psi_1(u) = ux$ and $x\psi_2(u) = xu$, for $x \in S_0$.

Hence $\psi_1(u)$ and $\psi_2(u)$ are respectively the left and right translations of $S_0$. Also $\psi_1(u)$ and $\psi_2(u)$ are linked since $x(uy) = (xu)y$for $u \in G$ and $x, y \in S_0$ Hence $(\psi_1(u), \psi_2\ (u)) \in \Omega(S_0)$ .Now define a function $\Psi$ from $G$ to $\Omega(S_0)$ as $\psi(u) = (\psi_1(u), \psi_2(u))$.Then clearly $\Psi$ is a homomorphism. Since $S$ is unit regular, for any $x \in S$ there is some $u \in G$ such that $xux = x$. Hence $ux$ and $xu$ belong to $E(S_0)$.Therefore $(x)\psi_1(u)$ [ or $(x)\ \psi_2\ (u)$) ]$\in E(S_0)$ for some $u \in G$.

Clearly $\psi_1(1)$ and $\psi_1(1)$ act as the identity permutations on $S_0$. Now for $u_1, u_2 \in G.$ and $x \in S_0$ $(u_1x)u_2 = u_1(xu_2)$. Therefore

$[(x)\psi_1(u_1)\ ]\ \psi_2\ (u_2)$ = $[(x)\psi_2(u_2)\ ]\ \psi_1\ (u_1)$ . So, $(x)$ $[\psi_1(u_1)\ \psi_2\ (u_2)]$ = $(x)$ $[\psi_2\ (u_2)\ \psi_1(u_1)\ ]$ for all $x \in S_0$. Therefore $\psi_1(u_1)\ \psi_2\ (u_2) = \psi_2\ (u_2)\ \psi_1(u_1)$ ,

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