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BUILDING THE GREAT PYRAMID BY LEVERING
A MATHEMATICAL MODEL

*H.J. de Haan**

*Professor emeritus, Delft University of Technology, Department of Geotechnology, h.jandehaan@orange.nl

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Note by the editor: *because of concerns about the legibility of the equations used in this contribution, we have opted to apply a one-column layout.*

ABSTRACT

A review of the extensive literature on the building of the Egyptian pyramids reveals that so far this problem has not been treated in a systematic, quantitative way. The present study aims at filling this gap by means of an integrated mathematical model, taking into account the interaction between various activities involved, such as quarrying, transportation and building. I focus my attention on the largest pyramid, the one built by Khufu.

The model simulates an efficient project co-ordination by balancing supply and demand of the building material, with all activities related to the growth of the pyramid and assuming a constant total workforce. This makes it possible to determine the effect of different building methods and of the productivity of the workers on the workforce required for the various tasks. In this paper only one building method has been considered, namely levering. Calculations have been carried out for two sets of input data, indicated as base case and maximum case.

Assuming a project duration of 20 years with 2624 working hours per year, the workforce for this building method is estimated to range from 4 000 to 10 000 men directly involved in the building of the pyramid and the supply of the necessary material.

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Introduction

Which methods were used and how many people were needed to build the large Egyptian pyramids is still a matter of debate. However, the extensive literature on the construction of the Egyptian pyramids is concerned mainly with architectural aspects. The few publications dealing with the building process consider only certain aspects in isolation and then usually in a semi-quantitative way[1]. To fill this gap an integrated mathematical model is proposed, taking into account the interaction between various activities involved, such as quarrying, transportation and building. The model has been used to compare different building methods in terms of workforce required.

From paintings it is evident that the Egyptians of the Old Kingdom made use of ramps and levers. Most egyptologists agree that these methods must have been used for the construction of the pyramids. In this paper I restrict myself to one building method namely levering[2]. Though not necessarily the most likely building method, this method was selected because no ramp is needed which simplifies the model. In this case only three activities have to be considered, namely quarrying, hauling the blocks from the quarry to the foot of the pyramid and transporting the blocks from the ground level to their position in the pyramid.

The model has been used to compare different building methods in terms of workforce required and to identify limitations, such as lack of working space on the ramp. Though simplified, the calculation procedure is essentially applicable to any other building method. To assess the sensitivity of the final results to the uncertainty in the basic data two sets of input data have been used, denoted as base case and maximum case. In fact, the main purpose of this paper is to relate the results to the basic data in a systematic, verifiable way, rather than to arrive at precise answers.

Approach

An important aspect of the building of these large pyramids is the high building rate. I assume the project took 20 years with all activities continuing throughout the year. Assuming one free day during the Egyptian ten-day week and eight effective working hours per day (excluding rest), this corresponds with 328 working days and 2624 working hours per year. For the Khufu pyramid this amounts to an average building rate of about 50 m^3/hour or, with an average block size of 1 m^3, roughly one block every minute. This means that the rate of supply of the building material must be tuned to the building rate in order to prevent major problems with accumulation or shortfall of building material. In this analysis the progress of all activities has therefore been related to the growth of the pyramid, while assuming a constant total workforce. For this purpose the volume of the pyramid has been determined in Appendix 1 as a function of its height. A constant total workforce means that the size of the teams in charge of the activities must be adjusted regularly. In view of the long project period concerned, this was probably not a serious constraint as there must have been enough time for training.

In this way I am, in fact, modelling a perfectly co-ordinated project, thus determining a minimum workforce. This may not be far from the truth, for when the Egyptians built the Khufu pyramid they could draw on the experience with similar projects of several generations before them. Therefore it seems likely that they went about their job in a co-ordinated way. In this respect, it is interesting to mention a more recent example of what experience can do. Whereas it took Dutch shipbuilders not more than eight months between 1627 and 1628 to build the 'Batavia', a ship meant for the East India trade, it took 1140 men 10 years between 1985 and 1995 to build its replica with better tools, but without the experience of their ancestors. Nevertheless, one has to

[1] Attempts have been made by Croon (1925), Illig & Löhner (1994), Isler (2001), Romer (2007), Smith (1999) and Wier (1996), but a critical review reveals that only the paper by Wier is based on a physical, though highly simplified model.
[2] A more general paper, also discussing building methods making use of ramps and considering a total of eight activities, is awaiting publication (De Haan, Forthcoming).

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bear in mind that in reality co-ordination is always more difficult and inevitably must have been less efficient than according to the ideal model presented here.

In the following chapters I shall estimate the output per man-hour, denoted as unit performance, for quarrying and transportation, as well as the resulting manpower (man-hours) and the workforce required. The basic data used are shown in table 1, the results obtained in table 2 and figures 1 to 5.

Unit performance for quarrying

Limestone

In the *NOVA* experiment (Lehner, 1997: 207), set up to simulate working conditions in ancient Egypt, 186 limestone blocks sized 1 m^3 were produced by 12 men in 22 days, corresponding with a unit performance of:

$$186 \times 1 / (22 \times 12 \times 8) = 0.089\ m^3/\text{man-hour}$$

Based on this experience Lehner estimates that under conditions in ancient Egypt production of the daily amount of about 322 m^3 would have required 1212 men, corresponding with:

$$322/ (1212 \times 8) = 0.033\ m^3/\text{man-hour}$$

Although the ancient Egyptians had a variety of stone and copper (alloy) tools at their disposal, such as chisels, saws and drilling tools, these could not of course, compete with the iron tools used in the NOVA experiment. On the other hand, as pointed out by Lehner (1997: 108), the core stones of the pyramid were probably less well cut than the outer stones, which would have speeded up the dressing. Moreover, the Egyptians had considerable experience with this type of work. For the base case and maximum case I assume:

$$u_3 = 0.03\ m^3/\text{man-hour and } u_3 = 0.02\ m^3/\text{man-hour}$$

Obviously, the Tura stone used to finish the pyramid, had to be dressed more carefully than the material from the Giza quarry which was to be used for the core. However, as this volume comprises only about 3 % of the total pyramid volume, this aspect can be neglected. Also the manpower involved in shipping this material from the Tura quarry to the building site was found to be of minor importance.

Granite

In the Khufu pyramid granite was used only for the King's chamber. This is only a small quantity (see table 1), but because of its hardness it required a relatively large quarrying effort. A rough estimate of the corresponding unit performance has been made based on the production process of Hatshepsut's obelisks which were reportedly produced and transported in seven months (Arnold, 1991: 40). An example is the unfinished obelisk at Aswan which gives a clear impression of the size of the trenches excavated around it and thus not only of the amount of material that had to be removed but also of the number of men the trenches could accommodate (*Ibidem*: 36-40; Goyon *et al.,* 2004: 164-166). Based on this information I arrived at a rate of removal of 0.00052 m^3/(man-hour). Engelbach (1923: 48), in an attempt to determine a value experimentally, found that he could remove a granite volume of 563 cm^3 in one hour (see also Goyon *et al.,* 2004: 164). Lehner (1997: 207) managed to remove 1800 cm^3 by pounding in five hours. This corresponds with a removal rate of 0.00056 and 0.00036 m^3/man-hour respectively, which is in reasonable agreement with the value derived above. For the much smaller blocks used for the burial chamber it is estimated that the volume to be removed is about twice the average block volume. The resulting unit performance in terms of useful product then becomes about 0.00052/2=0.00026 m^3/(man-hour). In spite of this extremely low unit performance, the manpower required for the quarrying turns out to be less than 1 % of the total manpower, due to the small volume involved. This aspect has therefore been neglected.

Unit performance for transportation

We assume that the building blocks were transported from the quarry to the building site on sledges and from there along the side faces of the pyramid by levering them from one step to the next. Below we shall derive the unit performance for these two modes of transportation.

Levering
I assume that the limiting factor for transportation is the maximum power that can be delivered per man. Levering along the sides of the pyramid is a process consisting of an alternation of vertical displacements controlled only by gravity and horizontal displacements controlled only by friction. If a block of volume V_1 and density ρ is moved by means of levers on two opposite sides by n_a men, with a lever ratio a, the forces per man required for vertical and horizontal displacement are respectively:

$$F_v = \frac{\rho g V_1}{a n_a} \text{ and } F_h = \mu \frac{\rho g V_1}{a n_a} = \mu F_v \tag{4-1}$$

Furthermore, I assume that F_v is equal to the maximum force F_m exerted per man and that the maximum power is exerted both for vertical and horizontal displacement. Using a lever results in a larger force on the block, but the velocities will be proportionally lower:

$$P_m = a F_m v_v = a \mu F_m v_h \tag{4-2}$$

According to (4-2):

$$\frac{v_v}{v_h} = \mu \tag{4-3}$$

The total time for one complete step, consisting of a horizontal displacement $L_p \Delta x$ and a vertical displacement $H \Delta z$, where Δx and Δz, are small fractions, then is

$$\Delta t = \Delta t_h + \Delta t_v = L_p \frac{\Delta x}{v_h} + H \frac{\Delta z}{v_v} = \frac{H \Delta z}{v_v} \left(\frac{\Delta x}{\Delta z} \frac{L_p}{H} \frac{v_v}{v_h} + 1 \right) = \frac{H \Delta z}{v_v} \left(\frac{\mu}{\tan \beta} + 1 \right)$$

where β is the slope of the side of the pyramid. Combination with (4-1) and (4-2) gives for the effective vertical velocity:

$$v_{v,eff} = \frac{H \Delta z}{\Delta t} = \frac{v_v}{1 + \mu / \tan \beta} = \frac{P_m n_a}{\rho g V_1} \frac{1}{1 + \mu / \tan \beta} \tag{4-4}$$

These velocities are maximum hauling velocities. To obtain the average speed one must take into account the time required for rest, for the return journey and to insert supports between successive 'jacks'. The velocities based on (4-2) therefore must be reduced by a retardation factor F_t to obtain the average velocities. For the sake of simplicity I assume that the retardation factor is the same for horizontal and vertical displacement. This factor then follows from:

$$F_t = \frac{v_{v,av}}{v_v} = \frac{v_{h,av}}{v_h} = \frac{P_{av}}{P_m} \tag{4-5}$$

Moreover, we have to introduce a factor F_n to correct for the number of men n_b inserting supports:

$$F_n = \frac{n_a}{n_a + n_b} \tag{4-6}$$

Thus, the unit performance – defined as the volume that can be displaced at a speed of 1 m/hour by

one man - for horizontal displacement, becomes according to (4-4) to (4-6):

$$u_{1h} = \frac{V_1}{n_a + n_b} v_{h,av} = \frac{F_{n1} F_{t1} P_m}{\rho\, g\, \mu} \tag{4-7}$$

Similarly, for vertical displacement, based on (4-4) to (4-7):

$$u_{1v} = \frac{V_1}{n_a + n_b} v_{v,eff,av} = \frac{F_{n1} F_{t1} P_m}{\rho\, g \left(1 + \mu / \tan\beta\right)} = \frac{u_{1h}}{1/\mu + 1/\tan\beta} \tag{4-8}$$

The average speeds $v_{v,\,av}$ and $v_{h,\,av}$ have been determined experimentally by Hodges (1989). Note that the unit performance does not depend on the size of the block.

Hauling

The velocity v_r is determined by the maximum power P_m and the force F exerted per man:

$$P_m = F\, v_r \tag{4-9}$$

The force required to move a block with volume V_1 and density ρ along an inclined surface with slope α and friction coefficient μ., is given by:

$$F_1 = \rho\, g V_1 \left(\sin\alpha + \mu \cos\alpha\right) \tag{4-10}$$

If F_m is the maximum force that can be exerted per man, the number of men required to move the block is:

$$n_a = \frac{F_1}{F_m} = \frac{\rho\, g V_1}{F_m} \left(\sin\alpha + \mu \cos\alpha\right) \tag{4-11}$$

The velocity v_r then is:

$$v_r = \frac{P_m}{F_m} = \frac{n_a P_m}{\rho\, g V_1} \frac{1}{\sin\alpha + \mu \cos\alpha} \tag{4-12}$$

For transport along a horizontal surface $\alpha = 0$, so that the unit performance becomes

$$u_{2h} = \frac{V_1 F_{2t} v_r}{n_a + n_b} = \frac{F_{2t} F_{2n} P_m}{\rho\, g\, \mu} \tag{4-13}$$

The unit performance corresponding with the vertical component of displacement then is[3]:

$$u_{2v} = \frac{V_1 F_{2t} v_r \sin\alpha}{n_a + n_b} = \frac{F_{2t} F_{2n} P_m}{\rho\, g \left(\mu / \tan\alpha + 1\right)} = \frac{u_h}{\left(1/\tan\alpha + 1/\mu\right)} \tag{4-14}$$

In this case n_b represents the number of men lubricating the surface and securing the transport by means of levers.

According to (4-10) $F_1 = 0$ and thus the block is in equilibrium if gravity and friction are balanced and thus

$$\mu = -\tan\alpha \tag{4-15}$$

[3] Strictly speaking, the vertical displacement corresponding with the depth of the quarry should be taken into account. As this effect adds little to the total manpower involved, it has been neglected.

The negative sign corresponds with a downwards, rather than upwards displacement. Based on this relationship the friction coefficient can be determined experimentally from the angle at which the sledge starts sliding down the ramp – or any other slope with the same combination of ramp and sledge surfaces – due to its own weight.

Numerical values

Number of workers per block

For the density of the building material a value of 2500 kg/m^3 was chosen, corresponding with the density of porous limestone (Arnold, 1991: 28, Table 2.1). For the maximum power per man Cotterell & Kamminga (1990, 39–41) mention a range from 70 to 100 W, corresponding with 252000 and 360000 Joule/man–hour, based on actual data, derived from 18th and 19th century experience with manual labour. I shall use a range from 200 to 300000 Joule/man–hour, corresponding with 83.3 and 55.5 W respectively. Furthermore, I assume that 20 to 30 kg or 200 to 300 N is a reasonable range for the maximum force per man, numbers which agree well with the experimental data reported by Hodges (1989) and Parry (2005). These forces cannot be realised precisely, however, as according to equations (4–1) and (4–11), they depend on the number of workmen occupied with the actual transportation of a block (*i.e.* excluding the helpers) which must be integers and in the case of levering an even number. The resulting number of men per team is shown in table 2 and the corresponding forces in table 1. By substituting these numbers, together with the number of helpers, namely two for levering and four for hauling, in (4–6) we find the corresponding correction factor.

Levering

With four men at the levers and two men inserting supports, Hodges (1989), using a lever ratio of 20, managed to lift a 2.5 ton load 813 mm in 200 seconds, and move it horizontally 190 mm in 20 seconds, corresponding with average speeds of 14.6 and 34.2 m/hour respectively. According to equation (4–3) and (4–5) this would imply a friction coefficient

$$\mu = \frac{v_{v,av}}{v_{h,av}} = \frac{14.6}{34.2} = 0.43$$

However, for the sake of consistency I use the same values for the friction factor for levering and for hauling, namely 0.25 and 0.5[4] for the base case and the maximum case respectively. I therefore accept the average vertical speed as determined by Hodges (1989) and derive the horizontal speed by dividing the vertical speed by the friction factor chosen: $v_{h,av}$ = 14.6/0.25 = 58.4 m/hour for the base case and $v_{h,av}$ = 14.6/0.5 = 29.2 m/hour for the maximum case.

In this case the retardation factor is taken to be equal to the ratio of the measured speed and the speed based on the maximum power per man. According to equations (4–1), (4–2) and (4–5):

$$F_t = \frac{v_{v,av}}{v_v} = v_{v,av}\frac{a\,F_m}{P_m} = v_{v,av}\frac{\rho\,g\,V_1}{n_a P_m} = 14{,}6\frac{2500\,x\,9.81\,x\,1}{4\,x\,300000} = 0.30$$

Time for loading is not applicable in this case. Moreover, in principle there is no downward traffic, because the blocks are passed on from one level to the next by means of a 'human chain'. As the number of builders increases (figure 4), additional levering teams must be added continually at the bottom and distributed over the side faces of the pyramid. Of course, workers may have to descend occasionally to rest. But even then their travel time is negligible compared to the average block travel time of more than three hours (table 2). The rather low retardation factor therefore must be due mainly to the time needed to rest, to insert the supports and to the careful, stepwise manoeuvring of the blocks, whereas the maximum speed corresponds with a perfect continuous

[4] For wood-on-wood in clean and dry condition (Handbook of Chemistry and Physics). As the Egyptians made use of lubrication by means of water or mud, this is probably a conservative estimate.

movement. In fact, as mentioned by Keable (in Hodges, 1989), with more experience the procedure might be speeded up, which means that the value used for the retardation factor may well be on the low side.

Hauling

Assuming that the total time on the job was at least 10 hours per day, I feel that the assumed work schedule of eight effective hours per day includes sufficient time for rest. The effect of the return trip depends on the transportation speeds. By substituting the values of P_m and F_m for the base case listed in table 1 in equation (4–9), we find a speed of about 980 m/hour, both for the base case and the maximum case. Assuming moreover that the hauliers return with their empty sledges at a speed of 5 km/hour, this implies a retardation factor

$$F_{t,return} = \frac{1}{1+980/5000} = 0.84$$

Bearing in mind that it took Hodges not more than 200 seconds to lift a 2.5 ton load 80 cm (see below), it is clear that loading and unloading the sledges could be done in a matter of minutes. Naturally, additional time was needed to fasten and unfasten the load. To be on the safe side, I assume 20 minutes[5] or 0.3 hour. The travel time from the quarry to the pyramid is 500/978 hours. The corresponding retardation factor is:

$$F_{t,loading} = \frac{500/980}{500/980+0.3} = 0.63$$

The combined retardation due to the return journey and loading and unloading then is

$$F_t = F_{t,return}\ x\ F_{t,loading} = 0.53$$

Using the same approach we find for the maximum case: $F_t = 0.65$. Substitution of the relevant data from table 1 and 2 in equations (4–7), (4–8) and (4–13) then results in the unit performances listed in table 2.

Comparison with ancient and experimental data

In literature several cases are reported of the transportation of heavy monuments. The only case where complete (though not necessarily reliable) data are available is that of the 'Green Naos', a 580 ton monument that was reportedly transported from Aswan to Sais, a distance of about 1000 km in a time of three years by 2 000 men. The force exerted per man including the friction factor is according to equation (4–10):

$$F_m = \frac{G}{n_a}\mu = \frac{9.81\,x\,580000}{2000}\mu = 2845\,\mu\ \ N/man$$

Obviously, the speed is an average, so that according to (4–9) – again with 2624 hours per year – the retardation factor follows from:

$$F_t = \frac{v_{av}}{v} = v_{av}\frac{F_m}{P_m} = \frac{1000000}{3\,x\,2624}\frac{F_m}{P_m} = 127\frac{F_m}{P_m}$$

Substituting values corresponding with our base case, namely F_m = 300 N and P_m = 300000 Joule/hour, we find μ = 0.105 and F_t = 0.127. These low values may well reflect that the transportation was done by a combination of levering and hauling, facilitated by lubrication, as

[5] That is twice as much as assumed by Dorka (2002), who considered the use of cranes for this purpose.

indeed shown in the scene depicting the transport of the monument of Djehutihotep (Newberry, 1895: pl. 15). In addition, a low retardation factor may be explained by the time needed for construction or repair of the road. Moreover, in view of the long distance, obstacles must have played a larger role, while the co–ordination of the large number of hauliers must have been a delaying factor as well.

Assuming that the helpers are included and thus F_n = 1, the unit performance becomes according to equation (4–13):

$$u_2 = \frac{F_n V v_{av}}{n_a} = \frac{F_n G v_{av}}{n_a \rho} = \frac{1 x 580000 x 127}{2000 x 2500} = 14.7\, m^3.m / man - hour$$

Unfortunately, I could not find any reliable experimental data on hauling experiments in the literature. The reason is that in all cases reported the force rather than the power per man is considered as the crucial factor. Little attention therefore was paid to the speed. As the maximum power per man plays a fundamental role in these calculations, I decided to perform an experiment myself. I am fortunate to have a 12 m long 'ramp' in my yard with a two–meter climb, covered with paving stones. The slope of the ramp corresponds with sin α=0.17 and cos α=0.985. The experiment consisted in pulling a couple of paving stones, piled on top of a wooden sledge up the ramp, while trying to deliver maximum power. I found that I could move a 35 kg (about 350 N) weight over a distance of 12.5 m in 14 seconds. The friction factor, determined according to (4–15), was found to be 0.48. According to (4-10) this corresponds with a force:

$$F_m = \rho g V_1 (\sin\alpha + \mu \cos\alpha) = 350 (0.17 + 0.48 x 0.985) = 225\, N$$

The power P_m exerted is:

$$P_m = F v_r = 225 \frac{12.5 x 3600}{14} = 225 x 3214 = 723000\ Joule / hour$$

To check the reliability the experiment was repeated several times and the results were found to be repeatable within 15 %. Of course the experiment lasted only seconds and maintaining this effort for a whole day would be a different matter. Nevertheless, in my view this demonstrates that the values assumed for P_m above are on the conservative side, bearing in mind also that the Egyptian workers were much better trained and that my estimates of the workforce required are based on the much lower average power P_{av}, corrected for the retardation effect.

Manpower

For quarrying the manpower depends only on the volume produced. As the unit performance u_3 is defined as the volume in m^3 that can be produced per man–hour, we have:

$$\frac{dM_3}{dV} = \frac{1}{u_3} \qquad (5–1)$$

Using the expression (1–3) derived for *dV/dz* in Appendix 1, we find:

$$M_3(z) = \int_0^z \frac{dM_3}{dV}\frac{dV}{dz}\,dz = 3\frac{V_p}{u_3}\int_0^z (1-z)^2\,dz = \frac{V_p}{u_3} z\left(z^2 - 3z + 3\right) \qquad (5–2)$$

For hauling the blocks from the quarry to the building site the manpower also depends on the distance L_2 covered:

$$\frac{dM_2}{dV} = \frac{L_2}{u_2} \tag{5–3}$$

Hence:

$$M_2(z) = \int_0^z \frac{dM_2}{dV}\frac{dV}{dz}\,dz = 3\frac{L_2\,V_p}{u_2}\int_0^z (1-z)^2\,dz = \frac{L_2\,V_p}{u_2}\,z\left(z^2-3z+3\right) \tag{5–4}$$

For pyramid building the blocks must first be lifted and subsequently moved horizontally to their final position in the pyramid. From the definition of the unit performance it follows that the manpower required to place a single block of volume V_1 at height h is given by:

$$\Delta M_1(z) = V_1\left(\frac{h}{u_{1v}} + \frac{\lambda L}{u_{1h}}\right) = V_1\left\{\frac{Hz}{u_{1v}} + \frac{\lambda L_p(1-z)}{u_{1h}}\right\} \tag{5–5}$$

where L is the width of the pyramid at height h and λL is the horizontal distance travelled by the block. The subscript v refers to the vertical component of the displacement along the side of the pyramid. The subscript h refers to the displacement along the horizontal top surface of the (truncated) pyramid.

We introduce the ratio c of the manpower corresponding with the horizontal and vertical displacement respectively. Combination with (4–8) and with

$$\frac{2H}{L_p} = \tan\beta$$

leads to:

$$c = \frac{\lambda L_p}{H}\frac{u_{1v}}{u_{1h}} = \frac{2\lambda/\tan\beta}{1/\mu + 1/\tan\beta} = \frac{2\lambda}{\tan\beta/\mu + 1} \tag{5–6}$$

Equation (5–5) can now be written:

$$\Delta M_1(z) = V_1\frac{H}{u_{1v}}\left\{(1-c)z + c\right\} \tag{5–7}$$

On the way to their final position in the pyramid the blocks follow different paths along the horizontal top surface of the pyramid and thus cover different distances, so that different values of λ would have to be used. To keep the calculation manageable, I assume that all blocks travel the same average distance λL. The calculation of λ is explained in Appendix 2.

By reducing V_1 and ΔM_1 to small increments dV and dM_1, (5–7) is reduced to:

$$\frac{dM_1}{dV} = \frac{H}{u_{1v}}\left\{(1-c)z + c\right\} \tag{5–8}$$

Integration gives:

$$\left.\begin{aligned} M_1(z) &= \int_0^z \frac{dM_1}{dV}\frac{dV}{dz}\,dz = 3\frac{H\,V_p}{u_{1v}}\int_0^z \left\{(1-c)z + c\right\}(1-z)^2\,dz \\ &= \frac{1}{4}\frac{H\,V_p}{u_{1v}}\,z\left\{3(1-c)z^3 - 4(2-3c)z^2 + 6(1-3c)z + 12c\right\} \end{aligned}\right\} \tag{5–9}$$

According to (5–2), (5–4) and (5–9), for z=1:

$$M_3(1) = \frac{V_p}{u_3} \tag{5–10}$$

$$M_2(1) = \frac{L_2 V_p}{u_2} \tag{5–11}$$

$$M_1(1) = \frac{V_p H}{4 u_{1v}}(1 + 3c) \tag{5–12}$$

By substituting the relevant data from tables 1 and 2, we find the total manpower for the three activities shown in table 2.

Workforce, building rate and time

Total workforce

In the previous chapter we determined the relationship between manpower required for the various activities and the relative height *z* of the pyramid. The number of men involved in a certain activity *i* at a given moment, denoted as *workforce*, equals the manpower in man–hours delivered per hour:

$$N_i(t) = \frac{dM_i(z)}{dt}$$

The total workforce then is:

$$N(t) = \frac{d}{dt}\{M_1(z) + M_2(z) + M_3(z)\} \tag{6–1}$$

As we aim at an optimum utilisation of the available workforce, we consider the case of a constant total workforce. This implies that the size of the three teams must be continuously adjusted, depending on the variation in manpower required for the corresponding activities. Since in (6–1) N(t) is assumed to be constant, the combined manpower is proportional to time, so that

$$N = \frac{M_1(z) + M_2(z) + M_3(z)}{t} = \frac{1}{F}\frac{M_1(1) + M_2(1) + M_3(1)}{T} \tag{6–2}$$

As the project life *T* is expressed in years and the manpower in man–hours, *T* must be multiplied by the number of working hours per year *F*.

Substitution in equation (6–2) of (5–10) to (5–12) gives:

$$N = \frac{V_p}{F\,T}\left[\frac{H}{4u_{1v}}(1 + 3c) + \frac{1}{u_3} + \frac{L_2}{u_2}\right] \tag{6–3}$$

Pyramid building rate

This rate follows from (6–1) in combination with equations (5–1), (5–3) and (5–8):

$$q(z) = \frac{dV}{dt} = \frac{\frac{d}{dt}\{M_1(z) + M_2(z) + M_3(z)\}}{\frac{d}{dV}\{M_1(z) + M_2(z) + M_3(z)\}} = \frac{N}{\frac{H}{u_{1v}}\{(1 - c)z + c\} + \frac{1}{u_3} + \frac{L_2}{u_2}} \tag{6–4}$$

In this case the factor *F* is not applicable, because *q* is supposed to be expressed in m^3/hour. The

average pyramid building rate is:

$$q_{av} = \frac{1}{F}\frac{V_p}{T} \tag{6–5}$$

Workforce by activity
Combination of (5–1), (5–3), (5–8) and (6–4) yields:

$$N_3(z) = \frac{dM_3}{dt} = \frac{dM_3}{dV}\frac{dV}{dt} = \frac{1}{u_3}q(z) \tag{6–6}$$

$$N_2(z) = \frac{dM_2}{dt} = \frac{dM_2}{dV}\frac{dV}{dt} = \frac{L_2}{u_2}q(z) \tag{6–7}$$

$$N_1(z) = \frac{dM_1}{dt} = \frac{dM_1}{dV}\frac{dV}{dt} = \frac{H}{u_{1v}}\{(1-c)z + c\}q(z) \tag{6–8}$$

Time
The relationship between the time (in hours) and z follows from the combination of (6–2) with equations (5–2), (5–4) and (5–9):

$$\left.\begin{aligned} t(z) &= \frac{M_1(z) + M_2(z) + M_3(z)}{N} \\ &= \frac{V_p}{N} z\left[\frac{1}{4}\frac{H}{u_{1v}}\left\{3(1-c)z^3 - 4(2-3c)z^2 + 6(1-3c)z + 12c\right\} + \left(\frac{1}{u_3} + \frac{L_2}{u_2}\right)\left(z^2 - 3z + 3\right)\right] \end{aligned}\right\} \tag{6–9}$$

To obtain the time in years t must be divided by F. By combining this equation with equation (1) of Appendix 1, the pyramid volume can be determined as a function of time (figure 1). The average time (in hours) to put in place a single block at level z – denoted as *average block delivery time* – equals its volume divided by the building rate:

$$t_d(z) = \frac{V_1}{q(z)} \tag{6–10}$$

Due to the small value of V_1, combined with the high building rate, which in turn is the result of $N_1(z)/(n_a + n_b)$ building teams working simultaneously, this is a very short time. The time needed to put in place a block by a single building team – denoted as *single block travel time* – thus is proportionally larger and by making use of (6–8) can be written:

$$t_t(z) = \frac{N_1(z)}{n_a + n_b}\frac{V_1}{q(z)} = \frac{H V_1}{u_{1v}}\frac{(1-c)z + c}{n_a + n_b} \tag{6–11}$$

This relationship follows also from the definition of the unit performance. The travel times corresponding with the bottom and the top of the pyramid are obtained by substituting in (6–11) $z=0$ and $z=1$ respectively. By averaging $t_t(z)$ over the pyramid volume, while taking into account equation (3) of Appendix 1 and equations (4–6), (4–8) and (5–6), the average block travel time is obtained:

$$\left.\begin{aligned} t_{t,av} &= \frac{V_1/V_p}{n_a + n_b}\frac{H}{u_{1v}}\int_0^1 \{(1-c)z + c\}\frac{dV}{dz}dz = \frac{3V_1}{n_a + n_b}\frac{H}{u_{1v}}\int_0^1 \{(1-c)z + c\}(1-z)^2 dz \\ &= \frac{1}{4}\frac{\rho g V_1 H}{F_t P_m n_a}(1+3c)(1+\mu/\tan\beta) = \frac{1}{4}\frac{\rho g V_1 H}{F_t P_m n_a}\left\{1 + \frac{\mu}{\tan\beta}(1+6\lambda)\right\} \end{aligned}\right\} \tag{6–12}$$

It is interesting to note that workforce, rate and time are all determined by the unit performances which in turn according to (4–7), (4–8) and (4–13) – apart from the correction factors – are uniquely determined by the maximum power per man. In contrast with the block delivery time the block travel time is independent of the block size, as for a given value of P_m the number of workers per block n_a is proportional to the block volume V_1.

Numerical results

Substitution of the relevant data from table 1 in the above equations then leads to the results shown in table 2 and figures 1 to 5. The building workforce increases (figure 4), in spite of a decreasing building rate (figure 3), because the rate effect is dominated by the effect of the increasing height (figure 2)[6]. The block travel and delivery times have been plotted in figure 5. According to equation (6–10) for a block size of 1 m^3 the blue curves are the inverse of the curves of figure 3 and according to equation (6–11) the black curves of figure 5 essentially represent the ratio of the curves of figures 4 and 3.

Initial and final workforce numbers for the three activities are summarised in the table below. As stated above, our model is based on the assumption of a perfect and optimum project organisation and thus, for a given set of basic data (table 1), leads to a lower limit for the workforce numbers. The maximum case, rather pessimistically, is based on the most unfavourable combination of the main variables (u_3, L_2, μ, P_m) as listed in table 1. Nevertheless, one cannot rule out the possibility that one or more of these are still underestimated. The transportation data are believed to be fairly reliable, as they are based on experimental and historical data related to manual labour, whereas the hauling distance is felt to cover an adequate range. This leaves the unit performance for limestone quarrying as the most uncertain factor. However, halving this parameter increases the upper limit of the workforce by not more than 27 %.

Workforce by activity

	base case		maximum case	
	start	end	start	end
quarrymen	2077	1025	2805	1785
hauliers	1295	639	5467	3480
builders	399	2107	958	3965
total	3771	3771	9230	9230

On the other hand, there are several reasons for believing that the numbers shown above are on the high side. As pointed out, the values used for the retardation factors, both for levering and for hauling may well be too low, the value used for the maximum power per man may be on the high side and according to some authors (Chevrier, quoted by Arnold, 1991: 63) the friction factor may be less than 0.25. Finally, the Egyptians may have been working more than eight hours per day.[7]

Some authors believe that the work was done on a seasonal basis. Presumably, this would mean that the number of working days per year would be reduced by a factor 3 or 4, thus multiplying the workforce requirements with the same factor. On the other hand, for a project life longer than the assumed 20 years the total workforce would be reduced proportionally.[8]

[6] Other authors, *e.g.* Romer (2007) and Smith (1999), apparently disregarding the effect of height, assume that the number of builders must decrease due to the smaller volumes of building material to be lifted. Borchardt (1932) and Isler (2001) make estimates based on a constant building rate.

[7] Romer (2007: 458, Table 3) assumes 10, Mahdy (2003) even 12 working hours per day.

[8] Edwards (1947: 123) and Stadelmann (1991: 227) think that Khufu may have reigned more than 20 years and that the pyramid perhaps was not even complete when he died as, according to some sources, his successor(s) completed the temple/pyramid complex. On the other hand, time was also needed for the planning and the preparation of the project.

The workforce figures shown represent only the workers involved in the actual supply and construction activities. In addition, there must have been a large supporting workforce for the fabrication and maintenance of tools and equipment and for the supply and preparation of food, possibly doubling the total number of people associated with the project.

Conclusions

Based on the assumption of a perfect project co–ordination a mathematical model has been developed that allows the determination of workforce requirements for the various activities involved and of the pyramid building rate, in a consistent and uniform way for different building methods and different basic data.

Assuming a constant total workforce, the number of workers required for the various activities changes as the project proceeds. The number of workers employed in the building of the pyramid increases with time and the number of those allocated to quarrying and hauling the building material to the building site, decreases accordingly.

Based on a total project life of 20 years with 2624 working hours per year and on a careful estimation of the other basic data, the workforce for the levering method was found to range from about 4 000 for the base case to 10 000 men for the maximum case.

During the project life the number of builders increases by a factor 5.3 and 4.1 for the base case and the maximum case respectively, whereas the building rate decreases by a factor 2.0 and 1.6 respectively. For the base case the time required to place a single block varies from 1.1 hour in the beginning to 11 hours to lift it to the top of the pyramid. For the maximum case these times vary from 2.1 to 14 hours. Due to the simultaneous effort of the large number of building teams the average time per block varies from about 1 to 2 minutes for both cases.

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Appendix 1: Pyramid Volume versus height

The pyramid is supposed to consist of an accumulation of horizontal layers, resulting in a succession of truncated pyramids of increasing height. The width of the top surface corresponding with a height h then is:

$$L = \left(1 - \frac{h}{H}\right) L_p = (1 - z) L_p$$

where z represents the relative height reached:

$$z = \frac{h}{H}$$

The volume of the truncated pyramid then equals the final volume minus the part remaining to be built:

$$V(z) = V_p \left\{ 1 - \left(\frac{H - h}{H} \right)^3 \right\} = V_p \left\{ 1 - (1 - z)^3 \right\} \qquad (1\text{–}1)$$

with $V_p = \frac{1}{3} H {L_p}^2$ (1–2)

Differentiation of (1–1) gives:

$$\frac{dV}{dz} = 3 V_p (1 - z)^2 \qquad (1\text{–}3)$$

Appendix 2: Calculation of the average horizontal travelling distance

This distance can be determined numerically by subdividing the horizontal area in n x n square elements. Let us assume that the blocks arrive on one side of the top surface of the pyramid at a distance $x=mL/n$ from a corner, then the distance a block must travel from that point to the centre of an arbitrary element $x=iL/n$, $y=jL/n$ is given by:

$$d_{ij} = \frac{L}{n}\sqrt{(i-m-0{,}5)^2 + (j-0{,}5)^2}$$

The average distance is obtained by adding the distances corresponding with all the elements i, j and dividing by their number n^2. Expressing the average distance as a fraction of the side L, we obtain:

$$\lambda = \frac{d_{av}}{L} = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{(i-m-0{,}5)^2 + (j-0{,}5)^2}$$

The blocks are supposed to be delivered at the midpoints of the four sides. This means that the surface can be subdivided in eight identical rectangular triangles, each containing $n^2/8$ elements, with the blocks arriving at the right angle: $m=n/2$. The diagonals connecting the corners of the top surface now are no–flow boundaries. To ensure that the elements on the diagonal are counted only once, we have to halve the corresponding distances, for which $j=i$:

$$\lambda = \frac{8}{n^3}\sum_{i=1}^{n/2}\left\{\sum_{j=1}^{i-1}\sqrt{(n/2-i+0{,}5)^2 + (j-0{,}5)^2} + \frac{1}{2}\sqrt{(n/2-i+0{,}5)^2 + (i-0{,}5)^2}\right\} \qquad (2\text{–}1)$$

Substitution of $n=20$ results in $\lambda = 0{,}271$. Doubling n changes this figure by less than 0,15 %.

Actually, the blocks arrive simultaneously at different points along the sides of the top surface, rather than all at the midpoint. This means that our calculation method leads to an over–estimation of the average distance covered.

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Appendix 3: Symbols and units used

a	lever ratio	
α	slope of ramp	
β	slope side of pyramid	
c	horizontal to vertical manpower ratio	
F	time schedule	hours/year
F_m	maximum force exerted per man	Newton
F_n	correction factor for helpers	
F_t	retardation factor	
g	gravitational acceleration	m/s^2
G	weight displaced	Newton
h	height reached	m
H	total height pyramid	m
L	length of pyramide side at height h	m
L_p	base of pyramide side	m
L_2	distance Giza quarry–building site	m
λ	average horizontal distance covered/side of pyramid	
μ	friction factor	
M_i	manpower for activity i	man–hour
N_i	workforce for activity is	
N	total workforce	
n_a	number of men per block pulling or levering	
n_b	number of helpers	
P_{av}	average power exerted per man	Joule/hour
P_m	maximum power exerted per man	Joule/hour
q	building rate	m^3/hour
ρ	density rock	kg/m^3
t	time	hours
T	total project duration	years
u_i	unit performance for activity i	*
v	velocity	m/hour
V_p	volume pyramid	m^3
V_1	block volume	m^3
z	h/H	

* m^3/manhour for quarrying, m^3.m/manhour for transport

Subscripts:

i	activity (1=levering, 2=hauling, 3=quarrying)
h	horizontal
v	vertical

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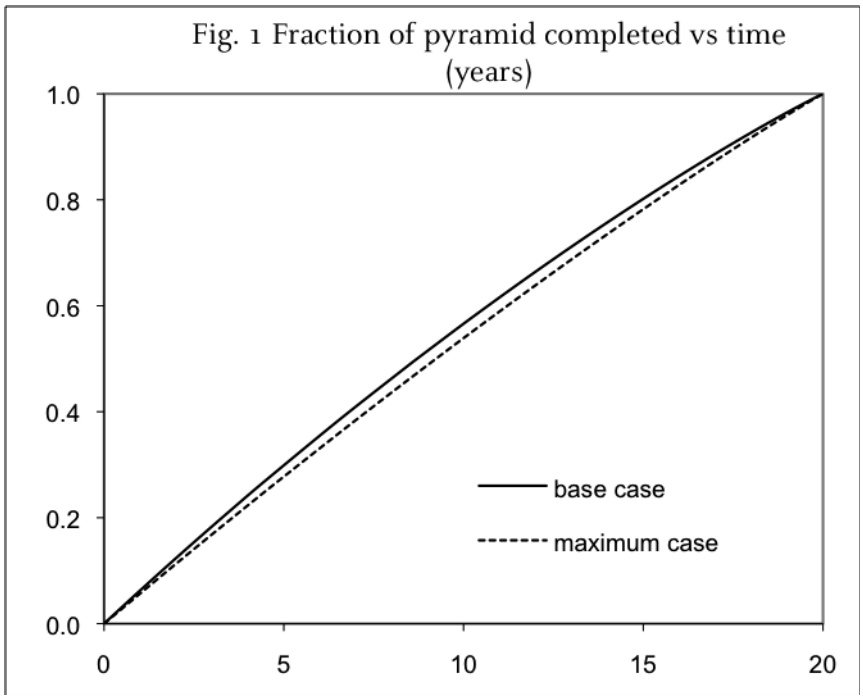
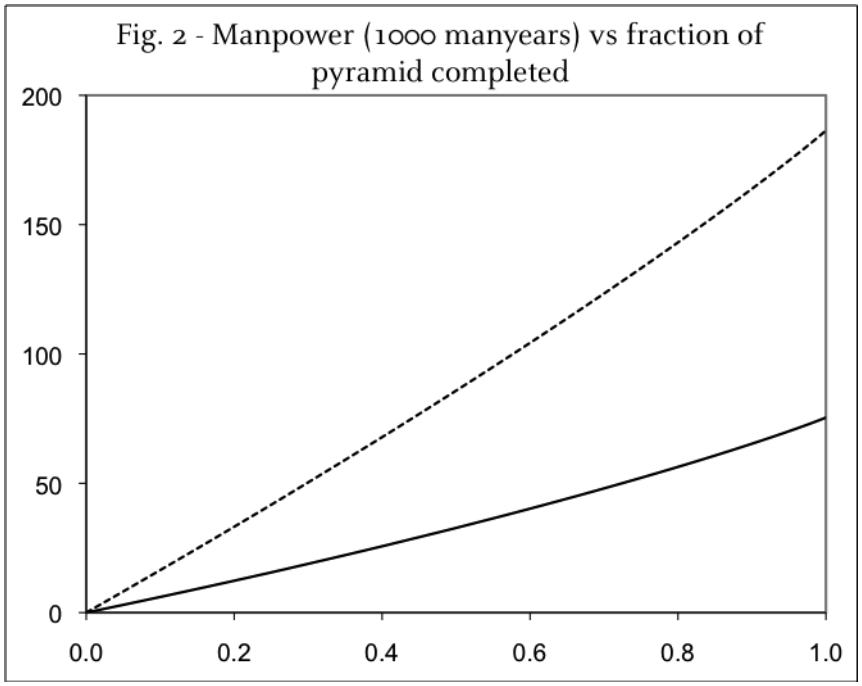
Appendix 4: Figures





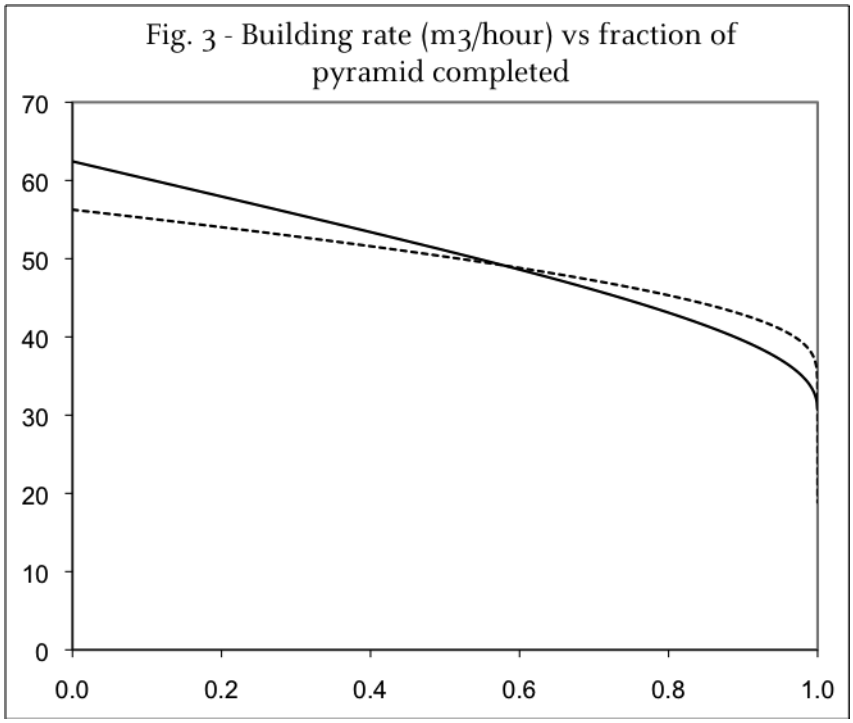


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Fig. 4 - Number of builders as a fraction of total workforce, vs fraction of pyramid completed

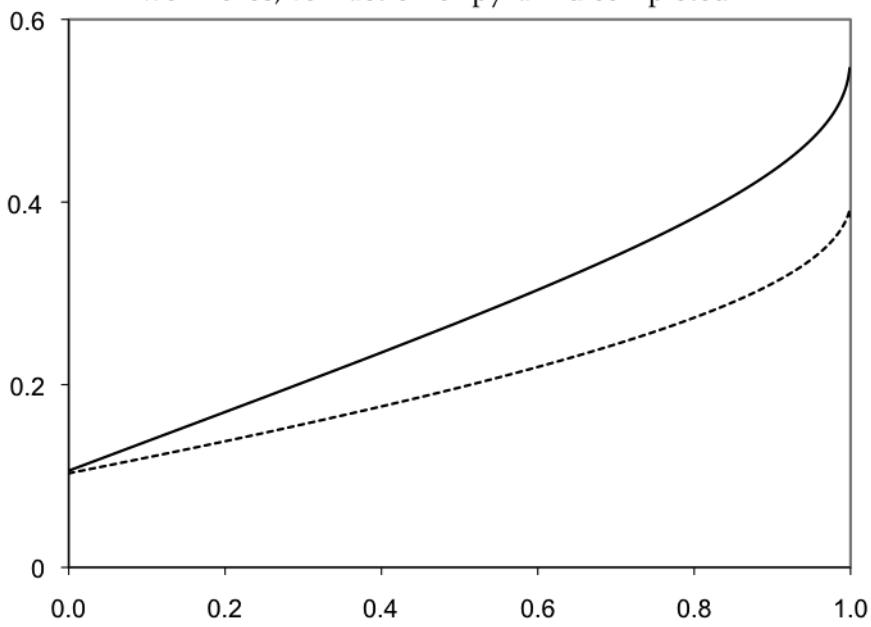
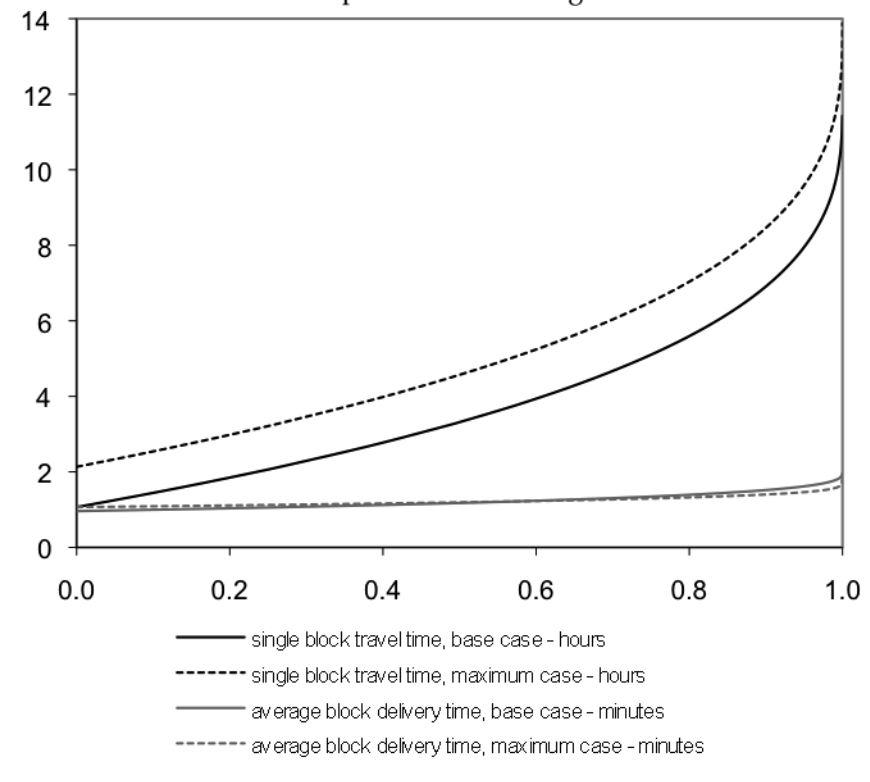


Fig. 5 - Block travel times vs fraction of pyramid completed for levering



Appendix 5: Tables

*Table 1 – Basic Data**

		unit
Times and Dimensions		
total project time	20	years
time schedule	2628	hours/year
height pyramid	146	m
base pyramid	230	m
average block volume	1	m^3
volume roofing blocks chamber	1159	m^3
Transportation		
rock density	2500	kg/m^3
maximum power per man**	300000–200000	Joule/man–hour
maximum force exerted per man**	306–204	N
number of helpers per block for levering	2	
number of helpers per block for hauling	4	
friction coefficient	0.25–0.50	
lever ratio	20	
average vertical levering speed***	14.6	m/hour
distance Giza quarry – building site	500–1000	m
Unit performances		
quarrying Giza limestone	0.03–0.02	m^3/man–hour
quarrying Aswan granite	0.00026	m^3/man–hour

* Where a range is given, the first (more favourable) figure refers to the base case, the second figure to the maximum case

** 1 N(ewton) = 0.102 kg 100000 N.m/hour =100000 Joule/hour = 27.8 W = 10194 kg.m/hour

*** Measured by Hodges and Keable

*Table 2 – Main Results Base Case**

	equation		unit
volume pyramid	1–2	2574467	m^3
Transportation			
number of men per block for levering (excl. helpers)**	4–1	4–6	
number of men per block for hauling (excl. helpers)**	4–11	28–72	
correction for number of helpers levering	4–6	0.67–0.75	
correction for number of helpers for hauling	4–6	0.93–0.97	
retardation factor levering	4–5	0.30	
retardation factor for hauling	4–5	0.53–0.65	
coefficient horizontal distance	2–1	0.27	
ratio manpower vertical/horizontal displacement	5–6	0.089–0.153	
average block travel time	6–12	3.79–5.09	hour
(average) block travel time at the bottom	6–11	0.22–0.77	hour
block travel time to the top	6–11	12.0–13.9	hour
Unit performances			
horizontal displacement levering	4–7	9.73–3.65	m^3.m/man–hour
horizontal displacement for hauling	4–13	24.1–10.3	m^3.m/man–hour
vertical displacement	4–8	2.03–1.31	m^3.m/man–hour
Manpower, workforce and building rate			
pyramid building	5–12	22.3–39.9	10^3 man–years
quarrying	5–10	32.7–49.1	10^3 man–years
hauling	5–11	20.4–95.6	10^3 man–years
total manpower		75.4–185	10^3 man–years
total workforce	6–2	3771–9230	men
average pyramid building rate	6–5	49.1	m^3/hour

* Where a range is given, the first (more favourable) figure refers to the base case, the second figure to the maximum case

** Adjusted to satisfy F_m (ca. 300 N/man for base case, ca. 200 N/man for maximum case, see Table 1)