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Slow Flow of Second-Order Fluid past a non-Newtonian Liquid Sphere under Stokes' Approximations

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ABSTRACT

The uniform steady slow viscous flow of an incompressible non-Newtonian second order fluid past a fixed sphere of Reiner-Rivlin fluid with constant coefficient of Newtonian-viscosities μ_{n_e} and μ_{n_i}, elastico-viscosity μ_e, and cross-viscosities μ_{c_e} and μ_{c_i} at very small Reynolds number has been delineated and discussed under the Stokes' approximation. In the present model, no-slip(slide) is assumed on the boundary of the fluid sphere's surface so as to satisfy continuity of shear across the surface. The unequivocal articulations for the stream functions are obtained to the second order in the small cross-viscous parameters S_e and S_i characterizing, respectively, the cross-viscosities of external and internal fluids, and special cases of flow past a solid and different fluid spheres are deduced then. The effects of forces exerted by external fluid in the flow on the streamlines and the drag on the fluid sphere have been investigated and also represented graphically. The effects of elastico-viscous parameter T on drag and different fluid parameters have also been studied.

1. Introduction

The steady translation of a axisymmetric particle, especially, spherical particle in a continuum hydrodynamics is an important problem in classical and fundamental fluid mechanics for the purpose of theoretical, computational and modeling point of views. Also, such hydrodynamics problems are base for the study and investigation of many physical, industrial, real-world applications. It was Stokes(1851) who initially solved the problem known as "falling-ball" for creeping viscous flow streaming over a sphere by undermining the inertia terms in comparison to viscous terms in the equation of motion. Further, Stokes calculated its terminal velocity by deriving analytical solution for both the fields, pressure and velocity.

In recent past, attempts have been made to clarify various non-theoretical outcomes in other than Newtonian liquids by the insertion of second order terms in the constitutive equation. But the governing equations of non-Newtonian fluid motions are harder than the Navier-Stokes' equations of motion and have rarely been unraveled except couple of cases. It appears to be practically difficult to acquire the solution of such equations in general solution when the motion is explained in 3-dimension. Also, when surrounding fluid has non-linear properties like cross-viscous, elastico-viscous, the situation becomes more complex as the flow governing equations are dominantly non-linear in nature. As such, it is impractical to use Stokes' method of linearization to study the effects of cross-viscosity in non-Newtonian fluids but only approximately either through a perturbation technique or numerical approach. There are ample examples available in literature for the problems of steady translation of spherical solid particle translating in an infinite expanse of fluid with uniform velocity. For decades, this unique problem has been a bench mark for researchers to evolve innumerable novel numerical techniques The problems of slow viscous motion of non-Newtonian fluids past a sphere is of much practical importance and is pretty useful in tribology, bio-fluid mechanics and petro-chemical industries like lubrication of bearings, lubrication of hip joints by means of synovial fluids etc. Jain (1955) acquired the solution for the slow flow of a non-Newtonian fluid by utilizing 'Synthetic Method'. Later, the same problem was revisited by Rathna (1962) using Stokes' approximation and expanding the stream function in powers of cross-viscous parameter S and analyzed that the drag experienced by a non-Newtonian liquid is equivalent to the Newtonian fluid with the same kinematical viscosity. But when the expansion is restricted to the second order, drag on the sphere was experienced higher as compared Newtonian fluid. A very identical approach was adopted earlier by Leslie(1961) to research the identical issue but for an Oldroyd visco-elastic fluid. Sharma (1979) examined the slow motion of a non-Newtonian second order fluid past a sphere. By means of a variational principle by Bird, Foster and Slattery(1963) found an approximate solution for creeping flow past a Reiner-Rivlin liquid sphere and applied variational principle to determine the drag coefficient. Rajagopal(1984) has discussed necessary and sufficient condition for the existence of solution under Stokes flow for second-order fluid. Saroa and Choudhury(1984) reinvestigated the problem by Rajagopal using the method of Blasius with the steady boundary layer flow

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Ramkissoon (1989b) solved the problem of slow flow of a Reiner-Rivlin liquid over a fluid sphere and inferred that sphere encounters more drag than classical fluid. Ramkissoon and Rahaman (2001) examined slow flow of a Reiner-Rivlin fluid in a fluid contained in solid spherical container. Ramkissoon (1999) examined the uniform flow of a polar fluid past a Reiner-Rivlin fluid sphere utilizing Stokes' estimate and determined drag experienced by the fluid sphere. Sahoo (2012) adopted a second order FDM to solve the steady B*o* dewadt flow of Reiner-Rivlin fluid and found that cross-viscous term decreases the radial velocity at significant distance from rotating disk. Recently, the problem of second-order fluid flow was studied applying magnetic field over torsionally oscillating disc by Agrawal and Agrawal (2016)

The prime goal for the present work is to look for how elastico-viscosity and cross-viscosity affects the flow phenomenon around a Reiner-Rivlin spherical fluid particle. The flow is considered to be uniform, slow viscous and steady. Although for Newtonian and some non-Newtonian fluid like micropolar fluid, couple stress fluid, etc. an exact solution is available in the literature, for a second-order fluid streaming over Reiner-Rivlin fluid under Stokesian approach this problem has not yet been studied earlier before. Present work is an extension of work by Sharma(1979) who considered rigid sphere instead fluid sphere.

2. Formulation of the Problem

A fluid sphere, having radius a of Reiner-Rivlin fluid whose rheological equation as suggested by Reiner (1945) is

$$T_{ij} = -P_{in}\,\delta_{ij} + 2\mu_{n_i}\,D_{ij} + 4\mu_{c_i}\,D_{ik}\,D_{kj}\,, \tag{1}$$

has been placed in a uniform stream of velocity U of a non-Newtonian second-order fluid whose constitutive equation as suggested by Coleman and Noll(1960) is given by

$$T_{ij} = -P_{ex}\,\delta_{ij} + 2\mu_{n_e}\,D_{ij} + +2\mu_{ev}\,E_{ij} + 4\mu_{c_e}\,D_{il}\,D_{lj}\,, \tag{2}$$

where

$$D_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right) \text{ and } E_{ij} = \frac{1}{2}\left(a_{i,j} + a_{j,i}\right) + U_{k,i}\,U_{k,j}\,, \tag{3}$$

T_{ij} is stress tensor, P_{ex} and P_{in} are respectively the hydrodynamic pressures for external and internal flow fields, U_i and a_i are velocity and acceleration vectors respectively, μ_{n_e} and μ_{n_i} are the Newtonian viscosities of external and internal fluids respectively, μ_{c_e} and μ_{c_i} are, respectively, the cross-viscosities of external and internal fluids, and μ_{ev} is the elastico-viscosity of external fluid.

For an incompressible fluid of density ρ, the flow governing equations in steady state for both the flow fields are, respectively, given by

$$T_{ij,j} = \rho U_k\,U_{i,k}\,, \tag{4}$$

$$U_{k,k} = 0, \tag{5}$$

where ρ represents the density of respective fluids.

We have utilized spherical polar co-ordinates (R, θ, ϕ) having the origin at the centre of the fluid sphere and $\theta = 0$ and $\theta = \pi$ are in the upstream and downward directions respectively(see Fig.1). Since the flow considered is axisymmetric $(\partial_\phi \equiv 0)$ in nature, so the parameters pertaining to flow fields are free from ϕ.

Let U_R and U_θ be velocity components along radial and transverse ($\perp$ radius of vector) directions respectively.

Hence, we can take velocity vector as

$$\boldsymbol{U}(R,\theta) = U_R\,\hat{e}_R + U_\theta\,\hat{e}_\theta = (U_R, U_\theta, 0)\,. \tag{6}$$

Now, to reduce the equations and quantities in non-dimensional form, utilizing the following transformations:

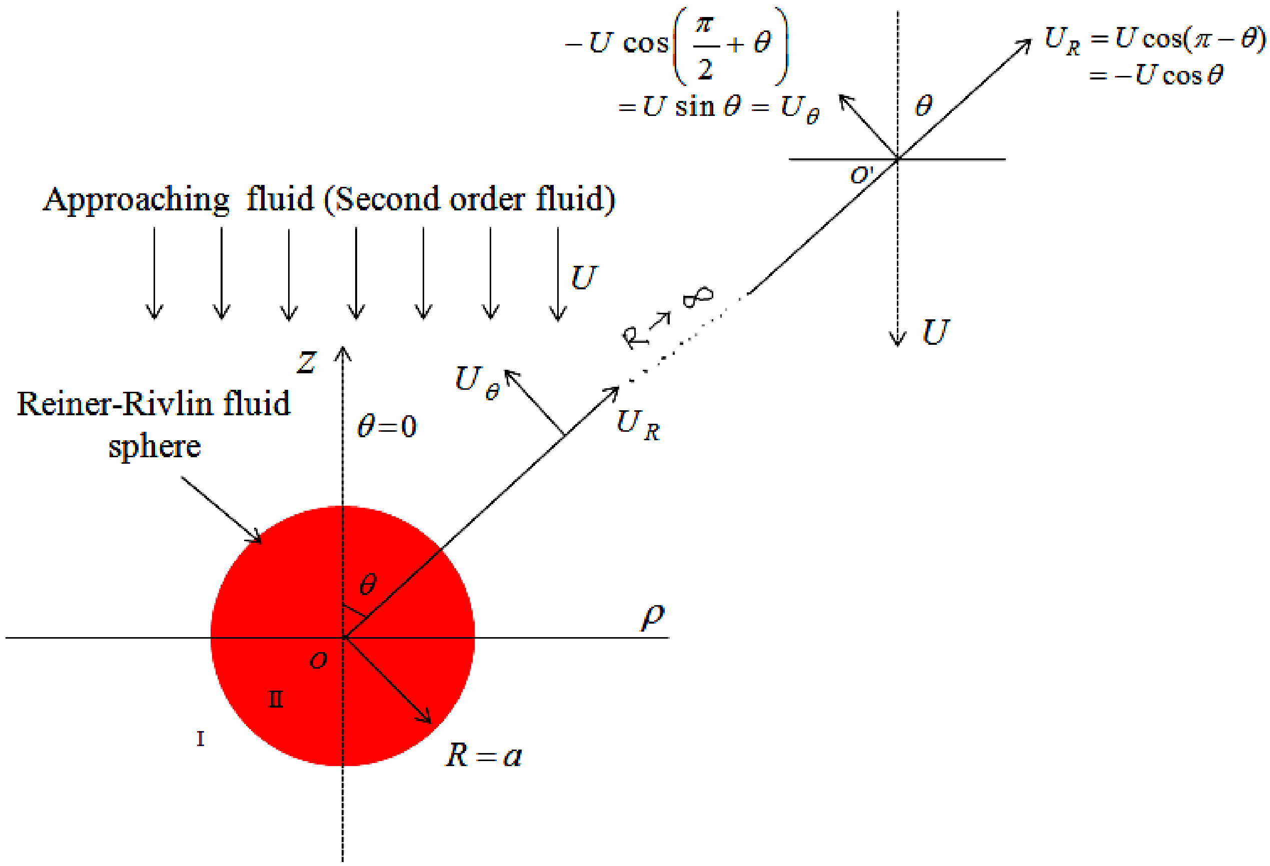

Figure 1. Physical delineation of non-Newtonian second order fluid model and coordinate system

$$\left.\begin{aligned} & r=\frac{R}{a},\ v_r=\frac{U_R}{U},\ v_\theta=\frac{U_\theta}{U},\ \psi=\frac{\Psi}{Ua^2},\ p=\frac{P}{\mu U/a} \\ & t_{ij}=\frac{T_{ij}}{\mu U/a},\ d_{ij}=\frac{D_{ij}}{U/a},\ e_{ij}=\frac{E_{ij}}{U^2/a^2},\ \lambda=\frac{\mu_{n_e}}{\mu_{n_i}}, \end{aligned}\right\} \tag{7}$$

where $r,\ v_r,\ v_\theta,\ \psi,\ p,\ t_{ij},\ e_{ij},\ d_{ij}$, ... stand for non-dimensional quantities, λ is relative viscosity, and a is radius of sphere and U is uniform velocity of external flow field.

Using non-dimensional quantities, Ramkissoon(1989a) has shown that the constitutive Eqs. (1) turns out to be in the components form as follows:

$$\left.\begin{aligned} t_{rr} &= -p + 2d_{rr} + 4S_i(d_{rr}^2 + d_{r\theta}^2),\\ t_{\theta\theta} &= -p + 2d_{\theta\theta} + 4S_i(d_{r\theta}^2 + d_{\theta\theta}^2),\\ t_{\phi\phi} &= -p + 2d_{\phi\phi} + 4S_i\, d_{\phi\phi}^2,\\ t_{r\theta} &= 2d_{r\theta} - 4S_i\, d_{r\theta}\, d_{\phi\phi}. \end{aligned}\right\} \tag{8}$$

While by Sharma(1979), the constitutive Eqs. (2) take the components form as follows:

$$\left.\begin{aligned} t_{rr} &= -p + 2d_{rr} + 2T e_{rr} + 4S_e(d_{rr}^2 + d_{r\theta}^2),\\ t_{\theta\theta} &= -p + 2d_{\theta\theta} + 2T e_{\theta\theta} + 4S_e(d_{r\theta}^2 + d_{\theta\theta}^2),\\ t_{\phi\phi} &= -p + 2d_{\phi\phi} + 2T e_{\phi\phi} + 4S_e\, d_{\phi\phi}^2,\\ t_{r\theta} &= 2d_{r\theta} + 2T e_{r\theta} - 4S_e\, d_{r\theta}\, d_{\phi\phi}. \end{aligned}\right\} \tag{9}$$

Whereas for both the regions, the momentum Eqs.(4) in terms of non-dimensional quantities yield the followings:

$$t_{rr,r} + r^{-1}t_{r\theta,\theta} + r^{-1}(2t_{rr} - t_{\theta\theta} - t_{\phi\phi} + t_{r\theta}\cot\theta) = Re\left(v_r v_{r,r} + r^{-1}v_\theta v_{r,\theta} - r^{-1}v_\theta^2\right), \tag{10}$$

$$t_{r\theta,r} + r^{-1}t_{\theta\theta,\theta} + 3r^{-1}t_{r\theta} + r^{-1}(t_{\theta\theta} - t_{\phi\phi})\cot\theta = Re\left(v_r v_{\theta,r} + r^{-1}v_\theta v_{\theta,\theta} + r^{-1}v_r v_\theta\right), \tag{11}$$

and the continuity Eqs.(5) are given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) = 0, \tag{12}$$

here $S_i (= \mu_{c_i} U / \mu_{n_i} a)$, $S_e (= \mu_{c_e} U / \mu_{n_e} a)$ denote the cross-viscous parameters for internal and external fluids respectively, and $T (= \mu_e U / \mu_{n_e} a)$ is elastico-viscous parameter for external fluid with $Re(= aU\rho / \mu_{n_i}$ or $aU\rho / \mu_{n_e})$ Reynolds numbers.

Now, due to axial symmetry, the velocity components of fluid flow in both the regions satisfying Eq.(12) can be expressed in terms of stream function ψ such that

$$v_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{13}$$

3. Validity of the Solution

The following assumptions have been taken into account for the sake of obtaining the valid solutions of Eqs. (10) and (11)

$$Re \ll (S_e, S_i, |T|) < 1 \quad \text{or} \quad a_2 \ll \nu_e, \quad U < a\nu_{n_e}/\nu_{c_e} \quad \text{and} \quad a_2 \ll \nu_{c_i}, \tag{14}$$

where ν_{c_i}, ν_{c_e}, ν_{n_i} and ν_e are kinematic viscosities of different fluids. Above conditions restrict the flow phenomenon like radius of the sphere, free-flow velocity, etc. and permit the neglect of inertia terms in comparison of viscous, cross-viscous and elastico-viscous terms as the motion is creeping in the sense of Stokes' approximation i.e. $Re \approx 0$. Hence, Eqs. (10) and (11) now concede

$$t_{rr,r} + r^{-1} t_{r\theta,\theta} + r^{-1}(2t_{rr} - t_{\theta\theta} - t_{\phi\phi} + t_{r\theta} \cot\theta) = 0, \tag{15}$$

$$t_{r\theta,r} + r^{-1} t_{\theta\theta,\theta} + 3r^{-1} t_{r\theta} + r^{-1}(t_{\theta\theta} - t_{\phi\phi}) \cot\theta = 0. \tag{16}$$

4. Statement and Solution of the Problem

We shall now explore the creeping steady flow of a non-Newtonian second order liquid streaming over a Reiner-Rivlin fluid sphere of radius $r = 1$. Also, assuming that the sphere is macroscopically at rest while the external fluid streams past it with uniform velocity U in the negative direction of the $z-$ axis in the absence of body forces and couples. Our ace motto is to find out drag on fluid sphere. Here, two fluid motions- external motion of second order fluid whose viscosity, elastico-viscosity and cross-viscosity to be designated by $(\mu_{n_e}, \mu_e, \mu_{c_e})$, and that of internal motion of Reiner-Rivin fluid with viscosity and cross-viscosity to be symbolized by (μ_{n_i}, μ_{c_i}). We distinguish between the separate fluid motions occurring outside and inside of the sphere by tagging on the subscripts ‘‘e’’ and ‘‘i’’, respectively.

Considering S_e and S_i small enough, we assume the following expressions for the quantities $\psi, v_r, v_\theta, t_{rr}, t_{r\theta}, \ldots,\ d_{rr}, d_{r\theta}, \ldots\ldots, e_{rr}, e_{r\theta}, \ldots\ldots$, and p :

$$\left.\begin{aligned} \Upsilon_e &= \Upsilon_e^{(0)} + S_e \Upsilon_e^{(1)} + S_e^2 \Upsilon_e^{(2)} + \ldots, \\ \Upsilon_i &= \Upsilon_i^{(0)} + S_i \Upsilon_i^{(1)} + S_i^2 \Upsilon_i^{(2)} + \ldots, \end{aligned}\right\} \tag{17}$$

where $\Upsilon_e^{(0)}, \Upsilon_e^{(1)}, \Upsilon_e^{(2)}, \ldots$ and $\Upsilon_e^{(0)}, \Upsilon_e^{(1)}, \Upsilon_e^{(2)}, \ldots$ can be regarded, respectively, as zeroth, first, second,..., appro- -ximations for external and internal fluid quantities as mentioned above. Also, S_e and S_i are cross-viscous parameters of external and internal fluids, respectively.

In case of creeping viscous flow past a solid sphere of radius $R = a\ (i.e.\ r = 1)$, it has already been proved by Sharma (1979) that if Eq.(17) is considered, then for external motion the stream function up to second order approximation in S_e yields the followings:

$$\left.\begin{aligned} \psi_e^{(0)} &= \left(r^2 - \frac{3}{2} r + \frac{1}{2r} \right) I_2(\zeta), \\ \psi_e^{(1)} &= 2\ell \left(1 - \frac{1}{r} \right)^3 I_3(\zeta), \\ \psi_e^{(2)} &= \ell \left[2 f_1(r) I_2(\zeta) + 4 f_2(r) I_2^2(\zeta) \right] + \\ &\quad + \alpha\ell^2 \left[2 g_1(r) I_3(\zeta) + 4 g_2(r) I_2(\zeta) I_3(\zeta) \right] \end{aligned}\right\} \tag{18}$$

where $\ell = \frac{3}{8}(1+\alpha)$, $\alpha = T / S_e$ and

$$f_1(r) = \frac{7}{80850}\left(132r - \frac{3469}{r} + \frac{51975}{r^2} - \frac{1413}{r^3} - \frac{95040}{r^3}\log r - \frac{40425}{r^4} - \frac{9075}{r^5} + \frac{2275}{r^7} \right),$$

$$f_2(r)=\frac{3}{308}\left(\frac{46}{r}-\frac{616}{r^2}+\frac{52}{r^3}+\frac{1056}{r^3}\log r+\frac{462}{r^4}+\frac{77}{r^5}-\frac{21}{r^7}\right),$$

$$g_1(r)=\left(36650.49-\frac{55753.97}{r^2}-\frac{4}{r^3}+\frac{1607.54}{r^4}+\frac{21.46}{r^5}-\frac{9.51}{r^5}\log r+\frac{17408.89}{r^6}+\frac{30.13}{r^7}+\frac{46.42}{r^8}-\frac{6.96}{r^9}\right),$$

$$g_2(r)=\left(-\frac{91.27}{r^2}+\frac{9}{r^3}+\frac{229.45}{r^4}-\frac{3.93}{r^4}\log r-\frac{51.6}{r^5}-\frac{210.96}{r^6}+\frac{177.03}{r^7}-\frac{72.23}{r^8}+\frac{10.58}{r^9}\right).$$

Particular solutions of ψ_e for external motion given by Eq.(18) vanish on sphere's surface at $r=1$, and $\psi_e \to r^2 I_2(\zeta)$ as r approaches to $\infty (i.e.\ r\to\infty)$. Furthermore, we also make a note of the followings that

$$f_1(1)=f_2(1)=g_1(1)=g_2(1)=0 \qquad (19)$$

and $f_1(r), f_2(r), g_1(r), g_2(r)\to 0$ as $r\to\infty$.

Similarly, It has already been shown by Ramkissoon (1989a) that if Eq.(17) is considered, then for internal motion of Reiner-Rivlin fluid within the liquid sphere the stream function up to second order approximation in S_i yields the followings:

$$\left.\begin{aligned}\psi_i^{(0)}&=2\left(r^4-r^2\right)I_2(\zeta),\\ \psi_i^{(1)}&=\frac{4}{21}r^5 I_3(\zeta),\\ \psi_i^{(2)}&=\frac{4}{63}r^6 I_2(\zeta).\end{aligned}\right\} \qquad (20)$$

In case of same second-order fluid past a liquid sphere, we can opt the external stream function ψ_e in the following given form

$$\psi_e=\psi_e^{(0)}+S_e\psi_e^{(1)}+S_e^2\psi_e^{(2)}+\sum_{n=2}^{\infty}\left(A_n r^{-n+1}+B_n r^{-n+3}\right) I_n(\zeta), \qquad (21)$$

where $\zeta=\cos\theta$ and $I_n(\zeta)$ are Gegenbauer functions. In particulars, the following identities in accordance with Legendre polynomials could be easily obtained [Happel and Brenner (1965)]

$$\left.\begin{aligned}I_2(\zeta)&=\frac{1}{2}(1-\zeta^2), & I_3(\zeta)&=\frac{1}{2}\zeta(1-\zeta^2),\\ I_4(\zeta)&=\frac{1}{8}(1-\zeta^2)(5\zeta^2-1), & I_5(\zeta)&=\frac{1}{8}\zeta(1-\zeta^2)(7\zeta^2-3),\\ I_2^2(\zeta)&=\frac{2}{5}I_2(\zeta)-\frac{2}{5}I_4(\zeta), & I_2(\zeta)\,I_3(\zeta)&=\frac{2}{7}I_3(\zeta)-\frac{2}{7}I_5(\zeta).\end{aligned}\right\} \qquad (22)$$

Utilizing the above cited relations (22) and stream function approximations (18), one can write stream function ψ_e, up to second order in S_e, for external motion illustrated by (21) as

$$\psi_e = \left(r^2 - \frac{3}{2} r + \frac{1}{2r} \right) I_2(\zeta) + 2\ell S_e \left(1 - \frac{1}{r} \right)^3 I_3(\zeta) + S_e^2 \Big[\ell \left\{ 2 f_1(r) I_2(\zeta) + 4 f_2(r) I_2^2(\zeta) \right.$$
$$\left. + \alpha \ell^2 \left\{ 2 g_1(r) I_3(\zeta) + 4 g_2(r) I_2(\zeta) I_3(\zeta) \right\} \Big] + \sum_{n=2}^{\infty} \left(A_n r^{-n+1} + B_n r^{-n+3} \right) I_n(\zeta). \tag{23}$$

Which, after a bit simplification, cedes the following

$$\psi_e = \left[r^2 + \left(B_2 - \frac{3}{2} \right) r + \left(A_2 + \frac{1}{2} \right) \frac{1}{r} + \frac{1}{5} \ell S_e^2 \left\{ 10 f_1(r) + 8 f_2(r) \right\} \right] I_2(\zeta) +$$
$$+ \left[2\ell S_e \left(1 - \frac{1}{r} \right)^3 + \frac{1}{7} \alpha \ell^2 S_e^2 \left\{ 14 g_1(r) + 8 g_2(r) \right\} + \frac{A_3}{r^2} + B_3 \right] I_3(\zeta) +$$
$$+ \left\{ -\frac{8}{5} \ell S_e^2 f_2(r) + \frac{A_4}{r^3} + \frac{B_4}{r} \right\} I_4(\zeta) + \left\{ -\frac{8}{7} \alpha \ell^2 S_e^2 g_2(r) + \frac{A_5}{r^4} + \frac{B_5}{r^2} \right\} I_5(\zeta) +$$
$$+ \sum_{n=6}^{\infty} \left(A_n r^{-n+1} + B_n r^{-n+3} \right) I_n(\zeta). \tag{24}$$

Also, it follows from the outcomes obtained by the authors Ramkissoon (1989a), Jaiswal and Gupta (2014a, 2014b, 2015), and later by Jaiswal (2017) that for internal motion within a Reiner-Rivlin liquid sphere, therefore the stream function ψ_i can be taken as given below

$$\psi_i = \psi_i^{(0)} + S_i \psi_i^{(1)} + S_i^2 \psi_i^{(2)} + \sum_{n=2}^{\infty} \left(C_n r^n + D_n r^{n+2} \right) I_n(\zeta), \tag{25}$$

where $\psi_i^{(0)}, \psi_i^{(1)}, \psi_i^{(2)}$ are given by (20). Inserting these values in (25) we acquire, up to the second order approximation, the following expression for stream function

$$\psi_i = \left\{ (C_2 - 2) r^2 + (D_2 + 2) r^4 + \frac{4}{63} S_i^2 r^6 \right\} I_2(\zeta) +$$
$$+ \left\{ C_3 r^3 + \left(D_3 + \frac{4}{21} S_i \right) r^5 \right\} I_3(\zeta) + \sum_{n=4}^{\infty} \left(C_n r^n + D_n r^{n+2} \right) I_n(\zeta). \tag{26}$$

5. Boundary conditions and determination of arbitrary constants

Anonymous coefficients emerging in (24) and (26) must be extricated from the following boundary conditions:

A. *Vanishing of normal at the interface needs that*

$$\psi_e = 0 \quad \text{on } r = 1, \tag{27}$$
$$\psi_i = 0 \quad \text{on } r = 1. \tag{28}$$

B. *Continuity of shear velocity across the interface requires that*

$$\frac{\partial \psi_e}{\partial r} = \frac{\partial \psi_r}{\partial r} \quad \text{on } r = 1, \tag{29}$$

C. *Continuity of shear stress is at the interface, i.e.,* $\left(t_{r\theta}\right)_e = \left(t_{r\theta}\right)_i$ *on* $r = 1$ *that yields*

$$\lambda \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi_e}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi_i}{\partial r}\right) \quad \text{on } r = 1, \tag{30}$$

where $\lambda = \mu_{n_e} / \mu_{n_i}$.

D. *Finally, we have*

$$\psi_e \to r^2 I_2(\zeta) \text{ as } r \to \infty. \tag{31}$$

The above stated boundary conditions lead, respectively, to the following set of non-homogeneous system of algebraic equations:

$$\left.\begin{array}{l}
A_n + B_n = 0 \quad \textit{for } n \geq 2, \qquad C_n + D_n = 0 \quad \textit{for } n \geq 4, \\
C_2 + D_2 = -\frac{4}{63} S_i^2, \qquad C_3 + D_3 = -\frac{4}{21} S_i, \\
-A_2 + B_2 - 2C - 4D_2 = 4 + \frac{8}{21} S_i^2, \\
-2A_3 - 3C_3 - 5D_3 = \frac{20}{21} S_i - 1.12\ell^2 \alpha S_e^2, \qquad -3A_4 - B_4 - 4C_4 - 6D_4 = 0, \\
-4A_5 - 2B_5 - 5C_5 - 7D_5 = 1.12\ell^2 \alpha S_e^2, \\
(n-1)A_n + (n-3)B_n + nC_n + (n+2)D_n = 0 \quad \textit{for } n \geq 6, \\
4\lambda A_2 - 2\lambda B_2 + 2Z_2 - 4W_2 = -3\lambda + 12 - \frac{2}{175} \ell \lambda S_e^2 + \frac{8}{7} S_i^2, \\
10\lambda A_3 - 10D_3 = -867167.54\ell^2 \alpha \lambda S_e^2 + 1.9048 S_i, \\
18\lambda A_4 + 4\lambda B_4 - 4C_4 - 18D_4 = \frac{12}{385} \ell \lambda S_e^2, \\
28\lambda A_5 + 10\lambda B_5 - 10C_5 - 28D_5 = -640.24\ell^2 \alpha \lambda S_e^2, \\
(n-1)(n+2)\lambda A_n + n(n-3)\lambda B_n - (n-3)nC_n + (n-1)(2+n)D_n = 0 \quad \textit{for } n \geq 6.
\end{array}\right\} \tag{32}$$

where $\lambda = \mu_{n_e} / \mu_{n_i}$. Solving above system of non-homogeneous equations yields

$$\left.\begin{aligned}
A_2 &= -B_2 = -\frac{0.00190476(262.5\lambda + 1\ell\lambda S_i^2 - 44.44S_i^2)\square_e}{1.+\lambda},\\
C_2 &= \frac{0.0634921(31.5+23.625\lambda - 0.03\ell\lambda S_i^2 + 2.33S_i^2 + 1\lambda S_i^2)\square_e}{1.+1.\lambda},\\
D_2 &= -\frac{0.1269841(15.75+11.8125\lambda - 0.015\ell\lambda S_i^2 + 1.67S_i^2 + \lambda S_i^2)\square_e}{1.+1\lambda},\\
A_3 &= -B_3 = -\frac{86716.75(-6.4578\times10^{-6}\ell^2\alpha S_e^2 + \ell^2\alpha\lambda S_e^2 - 4.39307\times10^{-11}S_i)\square_e}{1.+1.\lambda},\\
C_3 &= -\frac{86717.3(\ell^2\alpha\lambda S^2 - 4.39304\times10^{-11}S_i - 6.400801\times10^{-23}\lambda S_i)\square_e}{1.+1.\lambda},\\
D_3 &= \frac{86717.3(\ell^2\alpha\lambda S^2 - 2.19656\times10^{-6}S_i - 2.19652\times10^{-6}\lambda S_i)\square_e}{1.+1.\lambda},\\
A_4 &= -B_4 = C_4 = -D_4 = \frac{0.00222635\ell\lambda S^2\square_e}{1+\lambda},\\
A_5 &= -B_5 = -\frac{35.5689(0.015744\ell^2\alpha S_e^2 + 1\ell^2\alpha\lambda S_e^2)\square_e}{1+1\lambda},\\
C_5 &= -D_5 = -\frac{35.0089\ell^2\alpha\lambda S^2\square_e}{1+\lambda},\\
A_n &= 0 = B_n = C_n = D_n = 0 \qquad for\;\; n \geq 6.
\end{aligned}\right\} \tag{33}$$

6. Evaluation of drag force and some limiting cases

The most significant feature of the present flow problem is to evaluate hydrodynamic resisting force exerted on the Reiner-Rivin fluid sphere due to the flow of second-order fluid which can be attained by means of the formula [Payne and Pell(1960)]

$$F_D = 8\pi\,\mu_{n_e} \lim_{R\to\infty} \frac{\Psi_e(R,\zeta) - \Psi_\infty(R,\zeta)}{2R\, I_2(R,\zeta)}, \tag{34}$$

where $\Psi_\infty(R,\zeta)$ is the stream function corresponding to the fluid motion at infinity. The non-dimentional forms of both $\Psi_e(R,\zeta)$ and $\Psi_\infty(R,\zeta)$, respectively, are given by

$$\Psi_\infty(R,\zeta) = UR^2 I_2(\zeta),$$

$$\begin{aligned}\Psi(R,\zeta) &= Ua^2\Bigg[\left\{\left(\frac{R}{a}\right)^2 + \left(A_2+\frac{1}{2}\right)\frac{a}{R} + \left(B_2-\frac{3}{2}\right)\frac{R}{a} + \frac{1}{5}\ell S_e^2\left(10 f_1\left(\frac{R}{a}\right)+8 f_2\left(\frac{R}{a}\right)\right)\right\} I_2(\zeta) + \\ &+\left\{2\ell S_e\left(1-\frac{a}{R}\right)^3 + \frac{1}{7}\alpha\ell^2 S_e^2\left(14 g_1\left(\frac{R}{a}\right)+8 g_2\left(\frac{R}{a}\right)\right) + A_3\left(\frac{a}{R}\right)^2 + B_3\right\} I_3(\zeta) + \\ &+\left\{-\frac{8}{5}\ell S_e^2 f_2\left(\frac{R}{a}\right) + A_4\left(\frac{a}{R}\right)^2 + B_4\frac{a}{R}\right\} I_4(\zeta) + \\ &+\left\{-\frac{8}{7}\alpha\ell^2 S_e^2 g_2\left(\frac{R}{a}\right) + A_5\left(\frac{a}{R}\right)^4 + B_5\left(\frac{a}{R}\right)^2\right\} I_5(\zeta)\Bigg]. \end{aligned} \tag{35}$$

By implanting the expressions of $\Psi_e(R,\zeta)$ and $\Psi_\infty(R,\zeta)$ from Eq.(35) into the Eq.(34), we obtain

$$F_D = 8\,\pi\,\mu_{n_e}\, aU\,(-0.75 + 0.5B_2 + 0.0114286\,\ell S_e^2), \tag{36}$$

After inserting the value of B_2 from Eq.(33), one can get the required expression for drag as follows

$$F_D = \frac{8\,\pi\,\mu_{ne} aU}{1+\lambda}\{-0.75 - 0.5\lambda + \ell(0.0114286 + 0.012381\lambda)S_e^2 - 0.042328 S_i^2\}. \tag{37}$$

Some limiting cases:

(a). *When cross-viscous parameter* $S_e \to 0$ **:**

In this case second-order fluid becomes a Newtonian fluid and drag obtained is

$$F_D = -\frac{2a\pi U\left(9 + 6\lambda + \frac{32}{63}S_i^2\right)\mu_{n_e}}{3(1+\lambda)}. \tag{38}$$

Which agrees pretty well with the result obtained by Ramkissoon(1989).

(b). *When cross-viscous parameters for external and internal fluids i.e.* $S_e \to 0,\ S_i \to 0$ **:**

$$F_D = -\frac{6\,a\,\pi U\left(1+\frac{2}{3}\lambda\right)\mu_{n_e}}{1+\lambda}. \tag{39}$$

Again agrees good with the result obtained by Happel and Brenner(1965).

(c). *When elastico-viscous parameter* $T \to 0$ **:**

$$F_D = -6\, a\, \pi\, \mu_e U \left\{ \frac{1+\frac{2}{3}\lambda}{1+\lambda} + \left(\frac{-0.0057-0.0062\lambda}{1+\lambda} \right) S_e^2 + \frac{0.0564}{1+\lambda} S_i^2 \right\}. \quad (40)$$

This is very new result reported here for Reiner-Rivlin fluid dominating both the flow fields.

7. Discussions

The non-dimensional drag coefficient D_N is defined as the ratio of obtained drag force F_D exerted on Reiner-Rivlin fluid sphere to the drag force on a solid sphere which is a suitable approach in an unbounded expanse of fluid.

$$D_N = \frac{F_D}{-6\pi\mu_{n_e} aU}$$

$$= \frac{4}{3(1+\lambda)} \{0.75 + 0.5\lambda - \ell(0.0114286 + 0.012381\lambda)S_e^2 + 0.042328 S_i^2\}. \quad (41)$$

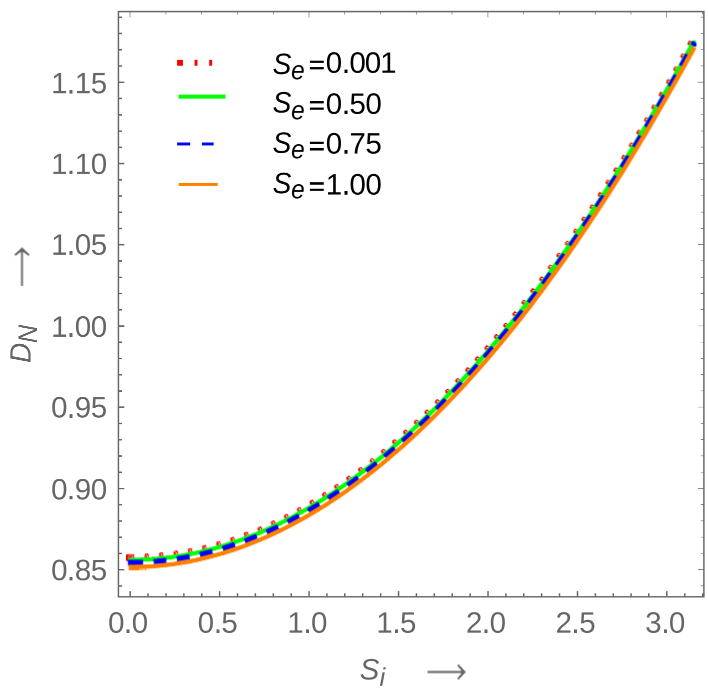

Figure 2. Variation of D_N with S_i for $T = -0.0086$, $\lambda = 0.75$ and various S_e.

The alteration in non-dimensional drag $\boldsymbol{D_N}$ w.r.t. different fluid parameters is shown graphically in Figures 2,3, 4 and 5.

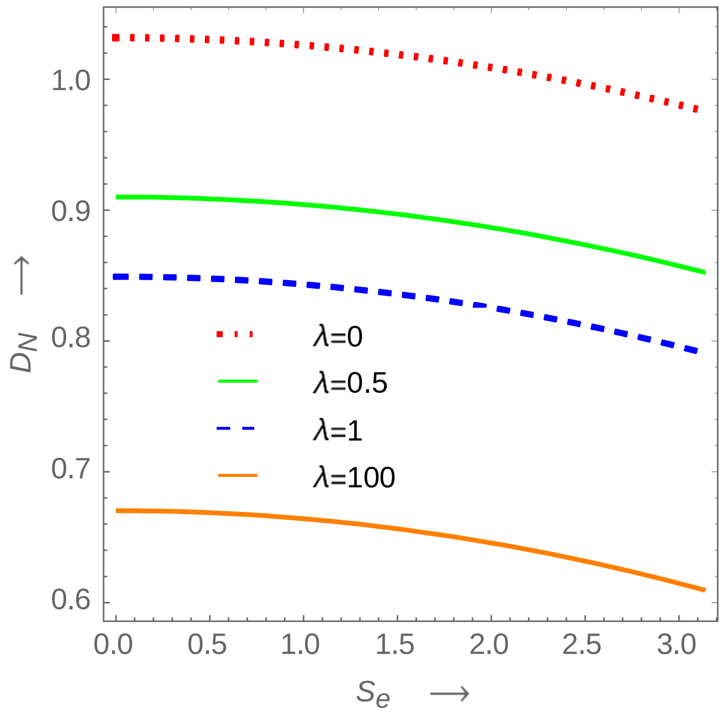

Figure 3. Variation of D_N with S_e for $T = -0.0086$, $S_i = 0.75$ and various λ.

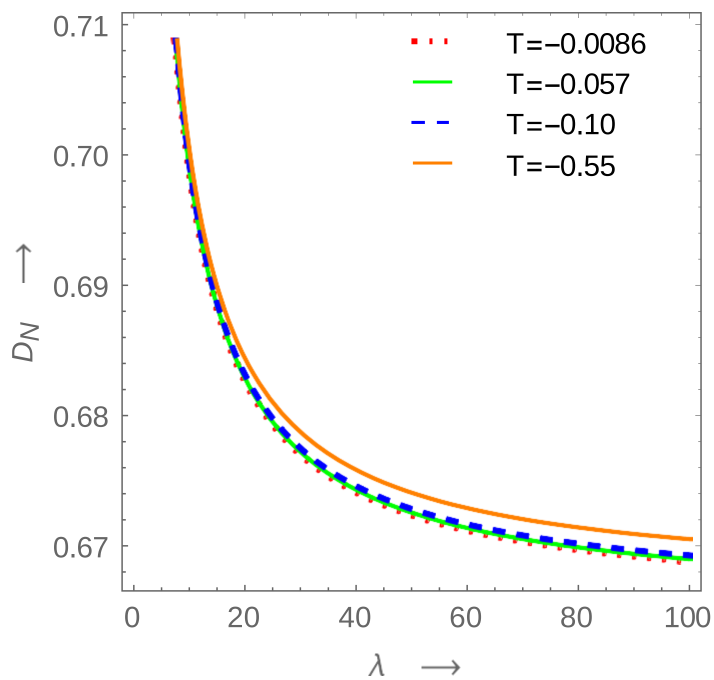

Figure 4. Variation of D_N with λ for $S_e = 0.5$, $S_i = 0.75$ and various T.

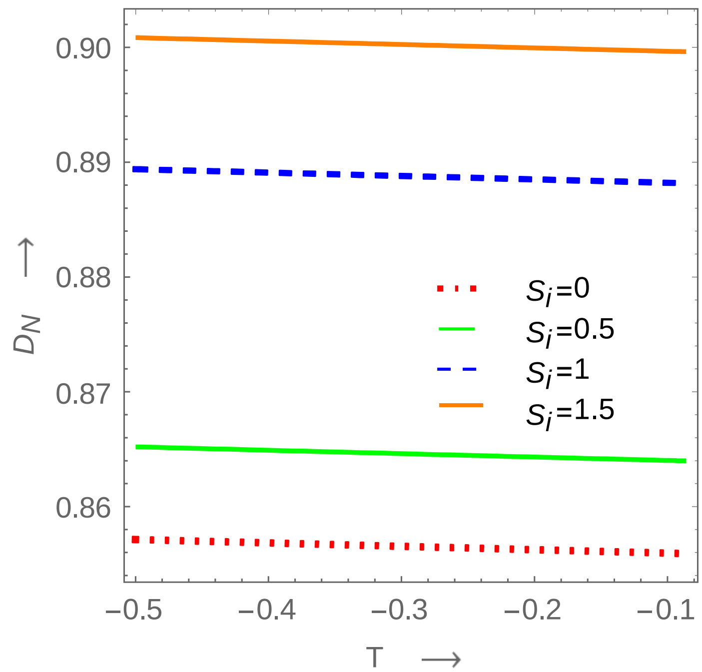

Figure 5. Variation of D_N with T for $S_e = 0.5$, $\lambda = 0.75$ and various S_i.

The alteration in $\boldsymbol{D_N}$ is plotted for various values of λ in Figure 3. It is observed that, at a fixed S_e, as viscosity ratio λ increases, drag decreases. It is due to the elastico-viscous effect on second-order fluid. The Figure 4 illustrates the drag coefficient $\boldsymbol{D_N}$ against relative viscosity λ. It shows that drag $\boldsymbol{D_N}$ decreases with decreasing value of T and λ which exhibits that liquid sphere experiences less drag as compared with solid case. It is depicted through the Figure 5 that keeping the remaining parameters unchanged, $\boldsymbol{D_N}$ is reported to be decreasing w.r.t. elastico-viscous. But this decrease in $\boldsymbol{D_N}$ is less for T as compared the increase in $\boldsymbol{D_N}$ for increasing S_i. Further, one can easily notice that drag $\boldsymbol{D_N}$ is less for Newtonian fluid sphere over Reiner-Rivlin fluid sphere.

8. Conclusions

The ace purpose pursued in this work is to investigate the flow of second-order fluid past a Reiner-Rivlin fluid sphere under the Stokes' approximation. Both the flow fields have been obtained in the form of stream functions expressing in a power series of S_e and S_i. Expression for the drag force evaluated in an analytical fashion yielding some renowned, novel and fruitful results as limiting cases validated with the past similar outcomes. The impact of numerous parameters such as Reynolds number *Re*, cross-viscous parameters S_e and S_i, elastico-viscous parameter T and relative viscosity λ on the flow is discussed and found to have a sturdy effect on the flow characteristics. The drag experienced on the sphere is examined. Furthermore, it is worth noticing that the external characteristics like cross-viscosity and elastico-viscosity diminish and internal cross-viscosity enhances the drag exerted on the implanted fluid sphere.

Conflict of Interest

Author himself declares that there is no competing interests for this publication.

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