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SIGNLESS LAPLACIAN ENERGY OF UNITARY ADDITION CAYLEY GRAPHS

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ABSTRACT

In 2011, bounds for the SignlessLaplacian energy of graphs and Unitary AdditionCayley graphs were introduced. In this paper, we determine theSignlessLaplacian eigenvalueand some bounds for the SignlessLaplacian energy of Unitary Addition Cayley graph.

1. Introduction

Let $G(V,E)$ be an undirected, simple , finite and connected graph with n vertices and m edges with vertex set $V(G)$ and edge set $E(G)$.The adjacency matrix of G is the $n\times n$ symmetric matrix $A(G)=\left(a_{ij}\right)$such that $a_{ij}=\begin{cases}1 & if\ v_i \& v_j\ are\ adjacent\\ 0 & otherwise\end{cases}$. The matrix $D(G)$of the graph G is a diagonal matrix of order n whose diagonal elements are the degree of the vertices in $V(G)$. Then the matrix $SLM(G)=A(G)+D(G)$is the $Signless\ Laplacian$ matrix [5].

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the $Signless\ Laplacian$ matrix are called $Signless\ Laplacian$ eigenvalues. The $spectrum$ of the $SLM(G)$is set of its eigenvalues together with their multiplicities. Then the $Signless\ Laplacian$ energy [5] is defined by

$SLM(G)=\sum_{i=1}^{n}\left|\lambda_i-\frac{2m}{n}\right|$.

For a positive integer $n>1$ the unitary Cayley graph X_n [1] is the graph whose vertex set $V(X_n)=Z_n=\{0,1,2,\dots(n-1)\}$and the edge set $E(X_n)=\{ab/a,b\,\epsilon\,Z_n,a-b\,\epsilon U_n\}$where $U_n=\{a\,\epsilon Z_n/\gcd(a,n)=1\}$.

The *Unitary Addition Cayley* graph $G_n(n>1)$ [3] is the graph whose vertex set

$V(G_n)=\{0,1,2,\dots,n-1\}$and the edge set $E(G_n)=\{ab\,/\,a,b\,\epsilon\,Z_n,a+b\,\epsilon\,U_n\}$,

Where,$U_n=\{a\,\epsilon\,Z_n/\,gcd(a,n)=1\}$.The graph G_n is regular if n is even and semiregular if n is odd. The number of edges of *Unitary Addition Cayley* graph is $\frac{n\varphi(n)}{2}$ if n is even and $\frac{n-1}{2}\varphi(n)$ if n is odd. The arithmetic function c(i,n) is a Ramanujan sum and is defined by

$c(i,n)=\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)}$,$t_i=\frac{n}{\gcd(i,n)}$ where $\varphi(n)$ denotes the Euler function and μ (n) denotes the mobius function.

In this paper we obtain the eigenvalues of *Signless Laplacian* matrix of *Unitary Addition Cayley* graph G_n. Also we compute the bounds for the *Signless Laplacian* Energy of *Unitary Addition Cayley* graph.

The upcoming sections follow the results and a detailed reference to [2, 3,4].

Result 1[1] The energy of unitary Cayley graph X_n is $2^r\varphi(n)$, r is the number of distinct prime factors dividing n.

Result 2 [4**]** Let A, A_1, A_2 be three real symmetric matrices of order n, such that A = A_1+ A_2. The eigenvalues satisfy the following inequalities:

for $1\le i\le n$and $0\le j\le \min\{i-1,n-i\}$, $\lambda_{i-j}(A_1)+\lambda_{1+j}(A_2)\ge\lambda_i(A)\ge\lambda_{i+j}(A_1)+\lambda_{n-j}(A_2)$.

2. Signless Laplacian Eigen values of Unitary Addition Cayley graph

The *Signless Laplacian* matrix of the *Unitary AdditionCayley* graph G_n is $SLM(G_n)=D(G_n)+A(G_n)$,where$A(G_n)$ is the adjacency matrix of G_n and $D(G_n)$ is the diagonal matrix whose diagonal elements are the degree of the vertices of the graph G_n.

The *Signless Laplacian* energy of a *Unitary Addition Caley* graph G_n is defined by $SLE(G_n)=\sum_{i=0}^{n-1}\left|\gamma_i-\frac{2m}{n}\right|$,where$\gamma_0,\gamma_1,\dots\gamma_{n-1}$are the *Signless Laplacian* eigen values of G_n.

Theorem 1 Let$n$ be even, then the *Signless Laplacian* eigenvalues for the *Unitary Addition Cayley*graph$G_n$is given by $\varphi(n)+\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)}$,$0\le i\le n-1$

Proof:

Consider the $Signless\ Laplacian$ matrix $SLM(G_n) = (a_{ij})$. $0 \leq i, j \leq n-1$.

$$(a_{ij}) = \begin{cases} 1 & if\ gcd(i+j, n) = 1, i \neq j \\ \varphi(n) & if\ i = j \\ 0 & otherwise \end{cases}$$

$SLM((G_n) = A(G_n) + \varphi(n)I, I$ is the identity matrix.

The eigenvalues of $A(G_n)$are $\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)}$, where $t_i = \frac{n}{\gcd(i,n)}$ where $\varphi(n)$ denotes the Euler function and $\mu(n)$ denotes the mobius function. The eigenvalues of φ(n)I are

$\varphi(n), 0 \leq i, j \leq n-1$

Then for the$Signless\ Laplacian$ eigenvalues of the $Unitary\ Addition\ Cayley$ graph G_n, where n is even is given by $\varphi(n) + \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)}$,$0 \leq i \leq n-1$.

Theorem 2 Let$n = p^m (m \geq 1)$,where p is prime. Then the $spectrum$ of the $Signless\ Laplacian\ Unitary\ Addition\ Cayley$ graph G_n is

$$\begin{pmatrix} \emptyset(n) - p^{m-1} - 2 & \emptyset(n) - 2 & \emptyset(n) & \emptyset(n) + p^{m-1} - 2 & \frac{x-y}{2} & \frac{x+y}{2} \\ \frac{p-3}{2} & (p-1)(p^{m-1}-1) & p^{m-1} - 1 & \frac{p-1}{2} & 1 & 1 \end{pmatrix}$$

Where, x = $2\varphi(n) + p^m - 2p^{m-1} - 2$ and y =$\sqrt{(p^m-2)^2 + 8p^{m-1}}$.

Proof:

Consider the adjacency matrix of G_n of order $k = p^{m-1}$ is $A(G_n) = \begin{bmatrix} A_1 & A_2 & \dots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & \vdots & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{bmatrix}$

Where,$A_1 = \begin{bmatrix} \varphi(n) & 1 & & 1 & \cdots & 1 \\ 1 & \varphi(n)-1 & & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & & \varphi(n)-1 \end{bmatrix}_{p \times p}$ and

$$A_2 = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 1 & 1 \end{bmatrix}_{p \times p}$$

Let $\bar{A}(G_n)$ is similar matrix of $A(G_n)$.

Then $\bar{A}(G_n) \begin{bmatrix} D_1 & J & & J & \cdots & J \\ J & D_1 + J - I & & J & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ J & O & J & \cdots & & D_1 + J - I \end{bmatrix}$ where

$$D_1 = \begin{bmatrix} \varphi(n) & 0 & \cdots & 0 \\ 0 & \varphi(n)-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi(n)-1 \end{bmatrix}, J = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$I$is an identity matrix and O is a null matrix of order k.

The Characteristic equation is $det(A(G_n) - \lambda I) = 0$. That is det $\bar{A}(G_n)$ - λI) = 0 $(\lambda - (\varphi(n) - k - 2))^{\frac{p-3}{2}}(\lambda - \varphi(n) + k - 2)^{\frac{p-1}{2}}(\lambda - \varphi(n))^{p^{m-1}-1}$
$\left(\lambda - (\varphi(n) - 2)\right)^{(p-1)(p^{m-1}-1)}[(\lambda^2 - \lambda(\text{kp} - 2\text{k} - 2 + 2\,\varphi(n)) + \varphi(n)(\text{kp} - 2\text{k} - 2 + 2\,\varphi(n)] = 0$.
The roots of the equation λ^2 - λ (kp - 2k – 2 + 2 $\varphi(n)$) + $\varphi(n)$(kp - 2k – 2 + 2 $\varphi(n)$) = 0 are $\lambda_1 = \frac{x+y}{2}$ and $\lambda_2 = \frac{x-y}{2}$, where x = $2\varphi(n) + p^m - 2p^{m-1} - 2$ and y $= \sqrt{(p^m - 2)^2 + 8p^{m-1}}$.
Thus the *spectrum* is

$$\begin{pmatrix} \emptyset(n) - p^{m-1} - 2 & \emptyset(n) - 2 & \emptyset(n) & \emptyset(n) + p^{m-1} - 2 & \frac{x-y}{2} & \frac{x+y}{2} \\ \frac{p-3}{2} & (p-1)(p^{m-1}-1) & p^{m-1} - 1 & \frac{p-1}{2} & 1 & 1 \end{pmatrix}$$

Where, x = $2\varphi(n) + p^m - 2p^{m-1} - 2$ and y=$\sqrt{((p^m - 2)^2 + 8p^{m-1}}$.

Corollary 3 Let $n = p\ (p > 3)$, where p is prime. Then the *spectrum* of *Signless Laplacian Unitary Addition Cayley*graph of G_n is $\begin{pmatrix} \varphi(n) - 3 & \varphi(n) - 1 & \frac{x-y}{2} & \frac{x+y}{2} \\ \frac{p-3}{2} & \frac{p-1}{2} & 1 & 1 \end{pmatrix}$, where x $=2\varphi(n) + p^m - 4$, and y=$\sqrt{((p^m - 2)^2 + 8}$.

Corollary 4 Let $n = 3$. Then the *Spectrum* of the *Signless Laplacian Unitary Addition Cayley* graph of $G_3$is $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

Theorem5 Let$n$ be odd. The *Signless Laplacian* eigenvalues of the *Unitary Addition Cayley*graphG_n is given by

$$\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - 2 \le \gamma_i \le \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n),\ 0 \le i \le \frac{n-1}{2}$$

$$-\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - 2 \le \gamma_i \le -\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n), \frac{n+1}{2} \le i \le n - 1.$$

Proof:
Consider the *Signless Laplacian* matrix $SLM(G_n) = (a_{ij}).\ 0 \le i,j \le n - 1$.

$$\text{Where,}a_{ij} = \begin{cases} \varphi(n) & if\ \gcd(i+j,n) \ne 1, i = j \\ \varphi(n) - 1 & if\ \gcd(i+j,n) = 1, i = j \\ 1 & if\ \gcd(i+j,n) = 1, i \ne j \\ 0 & otherwise \end{cases}$$

Split the given matrix $SLM(G_n)$ as $SLM(G_n) = B + C$
Where,$B = (b_{ij})$, $0 \le i,j \le n-1$, $b_{ij} = \begin{cases} 1 & if\ \gcd(i+j,n) = 1 \\ 0 & otherwise \end{cases}$ and

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$C = (c_{ij})$, $0 \le i,j \le n-1$, c_{ij} =

$$\begin{cases} \varphi(n) & if \gcd(i+j,n) \neq 1, i = j \\ \varphi(n)-2 & if \gcd(i+j,n) = 1 , i = j \\ 0 & otherwise \end{cases}$$

The eigenvalues of the left circulant matrix B are $(t_i)\frac{\varphi(n)}{\varphi(t_i)}$, $0 \le i \le \frac{n-1}{2}$ and

- $\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)}$,$\frac{n+1}{2} \le i \le n-1$.

The eigenvalues of the matrix C is $\varphi(n)$ with multiplicity $(n - \varphi(n))$ and $(\varphi(n) - 2)$ with multiplicity $\varphi(n)$.

On combining the eigenvalues of B and C together with the Result 2 we obtain the following.

$$\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - 2 \le \gamma_i \le \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n), \quad 0 \le i \le \frac{n-1}{2}$$

$$-\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - 2 \le \gamma_i \le -\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n), \frac{n+1}{2} \le i \le n-1.$$

3. Signless Laplacian Energy of Unitary Addition Cayley graph

Theorem 6: Let n be even .The $Signless\ Laplacian$ energy of $Unitary\ Addition\ Cayley$ graph G_n is $2^r\varphi(n)$, where r is the number of distinct prime factors of n.

Proof:

The $Signless\ Laplacian$ energy of $Unitary\ Addition\ Cayley$ graph G_n is

$$SLE(G_n) = \sum_{i=0}^{n-1} \left|\gamma_i - \frac{2m}{n}\right|$$

$$= \sum_{i=0}^{n-1} \left|\varphi(n) + \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} - \frac{2m}{n}\right|$$

$$= \sum_{i=0}^{n-1} \left|\varphi(n) + \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} - \frac{\frac{2n\varphi(n)}{2}}{n}\right|$$

$$= \sum_{i=0}^{n-1} \left|\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})}\right| = \varphi(n) \sum_{i=0}^{n-1} \frac{1}{\varphi(\frac{n}{\gcd(i,n)})}$$

$$= \varphi(n)2^r.$$

Theorem 7: Let $n = p^m, (m \ge 1)$,where p is prime. Then the$Signless\ Laplacian$ Energy of $Unitary\ Addition\ Cayley$ graph G_n is $SLE(G_n) =$

$$\begin{cases} \frac{10}{3} & if\ n = 3 \\ 2n - 5 - \frac{\varphi(n)(n-2)}{n} + \sqrt{(n-2)^2 + 8)} & if\ n = p \\ 3p^m - 4p^{m-1} - 2p - \frac{\varphi(n)}{n}(p^m - 2p^{m-1} - p + 1) + \sqrt{(p^m-2)^2 + 8p^{m-1}}, & if\ n = p^m, m \end{cases}$$

Proof:

The $Signless\ Laplacian$ energy of $Unitary\ Addition\ Cayley$ graph $G_n$is

$$\mathrm{SLE}(G_n) = \sum_{i=1}^{n} \left|\gamma_i - \frac{2m}{n}\right|$$

Case : (i) n = 3 , spectrum $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

$$\mathrm{SLE}(G_3) = \left|0 - \frac{4}{3}\right| + \left|1 - \frac{4}{3}\right| + \left|3 - \frac{4}{3}\right| = \frac{10}{3}$$

Case: (ii) $n = p, p > 3, spectrum = \begin{pmatrix} \varphi(n) - 3 & \varphi(n) - 1 & \frac{x-y}{2} & \frac{x+y}{2} \\ \frac{p-3}{2} & \frac{p-1}{2} & 1 & 1 \end{pmatrix}$

$$\mathrm{SLE}(G_n) = \left|\varphi(n) - 3 - \left(\frac{n-1}{n}\right)\varphi(n)\right| \times \left(\frac{p-3}{2}\right) + \left|\varphi(n) - 1 - \left(\frac{n-1}{n}\right)\varphi(n)\right| \times \left(\frac{p-1}{2}\right) + \left|\frac{x-y}{2} - \left(\frac{n-1}{n}\right)\varphi(n)\right| + \left|\frac{x+y}{2} - \left(\frac{n-1}{n}\right)\varphi(n)\right|$$

$$= \left(3 - \frac{\varphi(n)}{n}\right)\left(\frac{p-3}{2}\right) + \left(1 - \frac{\varphi(n)}{n}\right)\left(\frac{p-1}{2}\right) + \sqrt{(p-2)^2 + 8}.$$

$$= 2p - 5 - \frac{\varphi(n)}{n}(p-2) + \sqrt{(p-2)^2 + 8}$$

$$= 2n - 5 - \frac{\varphi(n)}{n}(n-2) + \sqrt{(n-2)^2 + 8}.$$

Case : (iii) $n = p^m, m > 1$ and $p \geq 3 spectrum$

$$\begin{pmatrix} \emptyset(n) - p^{m-1} - 2 & \emptyset(n) - 2 \blacksquare & \emptyset(n) & \emptyset(n) + p^{m-1} - 2 \blacksquare & \frac{x-y}{2} \blacksquare & \frac{x+y}{2} \\ \frac{p-3}{2} & (p-1)(p^{m-1}-1) \blacksquare & p^{m-1} - 1 & \frac{p-1}{2} \blacksquare & 1 \blacksquare & 1 \end{pmatrix}$$

$$\mathrm{SLE}(G_n) = \left|\varphi(n) - p^{m-1} - 2 - \left(\frac{n-1}{n}\right)\varphi(n)\right|\left(\frac{p-3}{2}\right) + \left|\varphi(n) - \left(\frac{n-1}{n}\right)\varphi(n)\right|(p^{m-1} - 1) + \left|\varphi(n) - 2 - \left(\frac{n-1}{n}\right)\varphi(n)\right|(p-1)(p^{m-1}-1) + \left|\varphi(n) + (p^{m-1} - 2)\left(\frac{n-1}{n}\right)\varphi(n)\right|\left(\frac{p-1}{2}\right)$$

$$+ \left|\frac{x-y}{2} - \left(\frac{n-1}{n}\right)\varphi(n)\right| + \left|\frac{x+y}{2} - \left(\frac{n-1}{n}\right)\varphi(n)\right|.$$

$$= \left(p^{m-1} + 2 - \frac{\varphi(n)}{n}\right)\left(\frac{p-3}{2}\right) + \left(2 - \frac{\varphi(n)}{n}\right)(p-1)(p^{m-1}-1) + \left(\frac{\varphi(n)}{n}\right)(p^{m-1} - 1) + \left(p^{m-1} - 2 + \frac{\varphi(n)}{n}\right)\left(\frac{p-1}{2}\right) + \sqrt{(p^m - 2)^2 + 8p^{m-1}}$$

$$= 3p^m - 4p^{m-1} - 2p - \frac{\varphi(n)}{n}(p^m - 2p^{m-1} - p + 1) + \sqrt{(p^m - 2)^2 + 8p^{m-1}}$$

Hence the result.

Theorem 8: Let n be odd and square free number, $n = p_1 p_2 \ldots p_r$ .The bounds for the *Signless Laplaian* energy of the *Unitary Addition Cayley* graph is

$$\frac{\phi(n)}{n}\ (1 + n\,2^r) - \frac{s+1}{2} \le SLE(Gn) \le \frac{\phi(n)}{n}\ (1 + n\,2^r) + \frac{2n-s-1}{2}$$

Where, s = $p_1, p_2, \ldots p_r$ be the maximal square free divisor of n.

Proof:

Consider the very sharp bounds of *Signless Laplacian* eigenvalues as

$$\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - \frac{s+1}{2} \le \gamma_i \le \mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) + n - \frac{s+1}{2}\ ,\ 0 \le i \le \frac{n-1}{2}$$

$$-\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) - \frac{s+1}{2} \le \gamma_i \le -\mu(t_i)\frac{\varphi(n)}{\varphi(t_i)} + \varphi(n) + n - \frac{s+1}{2},\ \frac{n+1}{2} \le i \le n-1.$$

The *Signless Laplacian* energy of G_n is $SLE(G_n) = \sum_{i=0}^{n-1}\left|\gamma_i - \frac{2m}{n}\right|$

First consider the lower bounds.

Energy of the lower bound = $\sum_{0\le i\le\frac{n-1}{2}}\left|\frac{\mu(\frac{n}{\gcd(i,n)})\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right| +$

$$\sum_{\frac{n+1}{2}\le i\le n-1}\left|-\frac{\mu(\frac{n}{\gcd(i,n)})\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right|$$

Consider $\frac{n}{\gcd(i,n)}$ as a square free number and divide each sum into two parts as $\mu\left(\frac{n}{\gcd(i,n)}\right) = 1$ if $\frac{n}{\gcd(i,n)}$ has an even number of distinct prime divisor and $\mu\left(\frac{n}{\gcd(i,n)}\right) = -1$ if $\frac{n}{\gcd(i,n)}$ has odd number of distinct prime divisors.

Hence, Energy of the lower bound =

$$\sum_{0\le i\le\frac{n-1}{2}}\left|\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right| + \sum_{0\le i\le\frac{n-1}{2}}\left|\frac{-\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right| \sum_{\frac{n+1}{2}\le i\le n-1}\left|-\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right| + \sum_{\frac{n+1}{2}\le i\le n-1}\left|\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right|$$

Therefore, Energy of the lower bound =

$$\sum_{0\le i\le\frac{n-1}{2}}\left[\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right] + \sum_{0\le i\le\frac{n-1}{2}}\left[\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} - \varphi(n) + \frac{s+1}{2} + \frac{n-1}{n}\varphi(n)\right] + \sum_{\frac{n+1}{2}\le i\le n-1}\left[\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} - \varphi(n) + \frac{s+1}{2} + \frac{n-1}{n}\varphi(n)\right] + \sum_{\frac{n+1}{2}\le i\le n-1}\left[\frac{\varphi(n)}{\varphi(\frac{n}{\gcd(i,n)})} + \varphi(n) - \frac{s+1}{2} - \frac{n-1}{n}\varphi(n)\right]$$

Energy of the lower bound = $\varphi(n)2^r + \varphi(n) - \frac{n-1}{n}\varphi(n) - \frac{s+1}{2}$

$=\frac{\varphi(n)}{n}(n2^r + 1) - \frac{s+1}{2}$

Similarly,

Energy of the upper bound = $\varphi(n)2^r + \varphi(n) - \frac{n-1}{n}\varphi(n) + n - \frac{s+1}{2}$

$$=\frac{\varphi(n)}{n}(n2^r + 1) + \frac{2n-s-1}{2}$$

Hence the bounds of the *Signless Laplacian* energy of G_n is

$$\frac{\phi(n)}{n}(1 + n\,2^r) - \frac{s+1}{2} \leq SLE(G_n) \leq \frac{\phi(n)}{n}(1 + n\,2^r) + \frac{2n-s-1}{2}.$$

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