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Another New Forms of Strongly and Quasi Mappings in Topological Spaces via δ-ß-Open Sets

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ABSTRACT

New kinds of generalized mappings namely, δ-ß-open, quasi δ-ß-open, and strongly δ-ß-open mappings in topological spaces are introduced and investigated by utilizing the concept of δ-ß-open sets. Several interesting characterizations and fundamental properties concerning of these forms of generalized mappings are obtained. Moreover, the relationships between such these of types of generalized mappings and other of well-known forms of generalized mappings are discussed.

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1. Introduction

1. Mappings and of course several generalized forms of open and closed mappings, strongly mappings and quasi mappings stand among the most significant notions and most researched points in the whole of mathematical sciences, and they have been introduced and investigated over the course of years. Certainly, it is hard to say whether one form is more or less important than another. Various interesting problems arise when one considers openness and closeness. Its importance is significant in various areas of mathematics and related sciences.

2. Recently, a number of generalizations of open sets have been considered such as: *b*-open sets, *b*-θ-open sets, *ß*-θ-open sets, *δ*-pre-open sets, *δ*-*ß*-open sets, *E*-open sets, *E*-θ-open, *δ*-*ß*-θ-open sets, E_c-open sets and δ-$ß_c$-open sets play an important role in generalization of continuity in topological spaces.
3. A class of generalized open sets in a topological space, called *δ*-*ß*-open sets or e^*-open sets was introduced and several of its fundamental and interesting properties were obtained by E. Hatir and T. Noiri [1] and Erdal E. [2] and introduced a new class of continuous mappings called *δ*-*ß*-continuous mappings into the field of topology.
In 1984, Rose [3] defined the notions of weakly open and weakly closed functions in topological spaces. In 2010, J. M. Mustafa [4] introduced some new generalized functions and investigated properties and characterizations of these new types of functions.
D. Sreeja and C. Janaki. [5] Introduced a new map called quasi πgb-closed map and investigated some of the fundamental properties of quasi πgb-irresolute functions.
In recent times, Generalized closed mappings and strongly closed mappings were introduced and studied, by M. L. Thivagar,…,etc [6], Furthermore, their relationships with various types of generalized closed maps are investigated. In addition, Alaa M. F. Al-Jumaili, in [**7**], introduced new concepts of generalized mappings called *E*-open, *E*-closed, quasi *E*-open, quasi *E*-closed, strongly *E*-open and strongly *E*-closed mappings in topological spaces by using the notion of *E*-open sets and explored their many interesting properties.In our paper, we will continue the study new forms of related mappings by involving *δ*-*ß*-open sets. The goal of the present this paper is to introduce and study several new types of open and closed mappings, strongly mappings and quasi mappings in topological spaces via *δ*-*ß*-open sets. Several interesting characterizations and some basic properties of these kinds of mappings are considered. Moreover, the relationships between of *δ*-*ß*-open, quasi *δ*-*ß*-open, and strongly *δ*-*ß*-open mappings and other well-known forms of generalized mappings are discussed.

2. PREREQUISITES

"Throughout the present paper$(\mathcal{X}, \mathcal{T}), (\mathcal{Y}, \mathcal{T}^*)$ and $(\mathcal{Z}, \mathcal{T}^{**})$(or simply $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset $\mathcal{A}$ of $\mathcal{X}$, The closure and interior of $\mathcal{A}$ are denoted by $Cl(\mathcal{A})$and $Int(\mathcal{A})$, respectively". We recall the following definitions of generalized open sets, which will be used often throughout our paper.

Definition 2.1: Let$(\mathcal{X}, \mathcal{T})$be atopological space. A subset $\mathcal{A}$ *of* $\mathcal{X}$ is said to be:

a) $\delta-$ open [8] if for each $x \in \mathcal{A}$ there exists a regular open set $\mathcal{V}$ such that $x \in \mathcal{V} \subseteq \mathcal{A}$. The $\boldsymbol{\delta}$-interior of $\mathcal{A}$ is the union of all regular open sets contained in $\mathcal{A}$ and is denoted by $Int_\delta(\mathcal{A})$. The subset $\mathcal{A}$ is called $\delta-$ open [8] if $\mathcal{A} = Int_\delta(\mathcal{A})$. A point $x \in \mathcal{X}$ is called a $\boldsymbol{\delta}$-cluster points of $\mathcal{A}$ [8] if $\mathcal{A} \cap \mathrm{Int}(\mathrm{Cl}(\mathcal{V})) \neq \varphi$, for each open set $\mathcal{V}$ containing x. The set of all δ-cluster points of $\mathcal{A}$ is called the $\boldsymbol{\delta}$-closure of $\mathcal{A}$ and is denoted by $Cl_\delta(\mathcal{A})$If $\mathcal{A} = Cl_\delta(\mathcal{A})$, then $\mathcal{A}$ is said to be $\delta-$ closed [8]. The complement of $\delta-$ closed set is said to be $\delta-$ open set. A subset $\mathcal{A}$ of a Topological space $\mathcal{X}$ is called $\delta-$ open [8] if for each $x \in \mathcal{A}$ there exists an open set $\mathcal{G}$ such that, $x \in \mathcal{G} \subseteq \mathrm{Int}(\mathrm{Cl}(\mathcal{G})) \subseteq \mathcal{A}$. The family of all $\delta-$ open sets in $\mathcal{X}$ is denoted by. $\delta\Sigma(\mathcal{X},\ \mathcal{T})$.

b) A subset $\mathcal{A}$ of a space $\mathcal{X}$ is called E-open [9] if $\mathcal{A} \subseteq \mathrm{Cl}(\delta - Int(\mathcal{A})) \cup \mathrm{I}nt(\delta - Cl(\mathcal{A}))$. The complement of an E-open set is called E-closed. The intersection of all E-closed sets containing $\mathcal{A}$ is called the E-closure of $\mathcal{A}$ [9] and is denoted by $E - Cl(\mathcal{A})$. The union of all E-open sets of $\mathcal{X}$ contained in $\mathcal{A}$ is called the E-interior [9] of $\mathcal{A}$ and is denoted by $E - \mathrm{Int}(\mathcal{A})$.

c) A subset $\mathcal{A}$ of a space $\mathcal{X}$ is called $\delta -$ ß $-$ open [1] or e*-open [2], if $\mathcal{A} \subseteq \mathrm{Cl}(\mathrm{Int}(\delta - \mathrm{Cl}(\mathcal{A})))$, the complement of a $\delta -$ ß $-$ open set is called, $\delta -$ ß $-$ closed. The intersection of all $\delta -$ ß $-$ closed sets containing $\mathcal{A}$ is called the $\delta -$ ß $-$closure of $\mathcal{A}$ [1] and is denoted by $\delta -$ ß $- \mathrm{Cl}(\mathcal{A})$. The union of all $\delta -$ ß $-$ open sets of $\mathcal{X}$ contained in $\mathcal{A}$ is called the $\delta -$ ß $-$interior [1] of $\mathcal{A}$ and is denoted by $\delta -$ ß $- Int(\mathcal{A})$.

4. **Remark 2.2:** The family of all E-open (resp. E-closed, $\delta -$ ß $-$ open, $\delta -$ ß $-$ closed) subsets of $\mathcal{X}$ containing a point $x \in \mathcal{X}$ is denoted by $E\Sigma(\mathcal{X},\ x)$ (resp. $EC(\mathcal{X},\ x)$, $\delta -$ ß$\Sigma(\mathcal{X},\ x)$, $\delta -$ ß$C(\mathcal{X},\ x)$). The family of all E-open (resp. E-closed, $\delta -$ ß $-$ open, $\delta -$ ß $-$ closed) sets in $\mathcal{X}$ are denoted by $E\Sigma(\mathcal{X},\ \mathcal{T})$ $(resp. EC(\mathcal{X},\ \mathcal{T})$, $\delta -$ ß$\Sigma(\mathcal{X},\ \mathcal{T})$, $\delta -$ ß$C(\mathcal{X},\ \mathcal{T}))$.

5.

6. **Proposition 2.3**: [**9**, **10**] the following properties hold for a space $\mathcal{X}$:

a) The Arbitrary union of any family of $E - (resp.\ \delta -$ ß$) -$ open sets in $\mathcal{X}$, is an $E - (resp.\ \delta -$ ß$) -$ open set.

b) The Arbitrary intersection of any family of $E - (resp.\ \delta -$ ß$) -$ closed sets in $\mathcal{X}$, is an $E - (resp.\ \delta -$ ß$) -$ closed set.

Definition 2.4: Let $(\mathcal{X},\ \mathcal{T})$ be a topological space. A subset $\mathcal{A}$ of $\mathcal{X}$ is said to be:

a) Regular open (resp. regular closed) [**11**] if $\mathcal{A} = Int(Cl(\mathcal{A}))$ (resp. $\mathcal{A} = Cl(Int(\mathcal{A}))$).
b) α-open [**12**], if $\mathcal{A} \subseteq Int(Cl(Int(\mathcal{A})))$, and $\alpha -$ closed if $Cl(Int(Cl(\mathcal{A}))) \subseteq \mathcal{A}$.
c) semi-open [**13**], if $\mathcal{A} \subseteq Cl(Int(\mathcal{A}))$, and semi $-$ closed if $Int(Cl(\mathcal{A})) \subseteq \mathcal{A}$.
d) pre-open [**14**], if $\mathcal{A} \subseteq Int(Cl(\mathcal{A}))$, and pre $-$ closed if $Cl(Int(\mathcal{A})) \subseteq \mathcal{A}$.
e) *ß*-open [**15**], if $\mathcal{A} \subseteq Cl(Int(\mathrm{Cl}(\mathcal{A})))$, and ß $-$ closed if $Int(Cl(Int(\mathcal{A}))) \subseteq \mathcal{A}$.
f) *b*-open [**16**]) if $\mathcal{A} \subseteq Int\big(Cl(A)\big) \cup Cl(Int(A)))$, and b $-$ closed if $Int\big(Cl(A)\big) \cap Cl(Int(A))) \subseteq \mathcal{A}$.
Remark 2.5: "From definitions **(2.1)** and **(2.4)** we have the following figure in which the converses of implications need not be true, see the examples in [**10**], [**9**] and [**2**]".

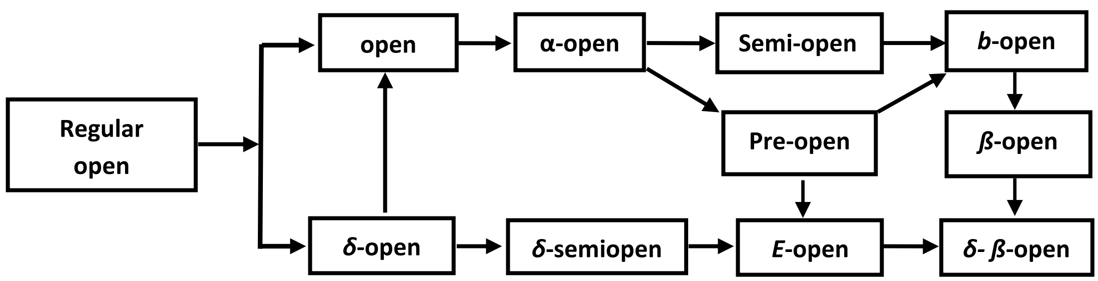

Figure (1): The relationships among some well-known generalized open sets in Topological spaces

CHARACTERIZATIONS OF $\delta -$ ß $-$ OPEN MAPPINGS

This section is devoted to introduce several characterizations and some properties concerning of $\delta -$ ß $-$ open mappings by utilizing $\delta -$ ß $-$ open sets .
Definition 3.1: A mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is said to be $\delta -$ ß $-$ open if
$f(\mathcal{U}) \in \delta -$ ß$\Sigma(\mathcal{Y}, \mathcal{T}^*)$ for every open set $\mathcal{U}$ in $\mathcal{X}$.
Theorem 3.2: A mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is $\delta -$ ß $-$ open iff $\forall x \in \mathcal{X}$ and each open set $\mathcal{U}$ in $\mathcal{X}$. with $x \in \mathcal{U}$, there exists a set $\mathcal{V} \in \delta -$ ß$\Sigma(\mathcal{Y}, \mathcal{T}^*)$ containing $f(x)$ such that, $\mathcal{V} \subseteq f(\mathcal{U})$.
Proof: The proof is follows directly from definition **(3.1).**

Theorem 3.3: Let $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$be $\delta - ß -$ open. If $\mathcal{V}\ \subseteq$ $\mathcal{Y}$ and $\mathcal{M}$ is a closed sub set of $\mathcal{X}$ containing$f^{-1}(\mathcal{V})$, then $\exists\ \mathcal{F} \in \delta -$ ßC$(\mathcal{Y},\ \mathcal{T}^*)$containing $\mathcal{V}$ (s.t) $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$.

Proof: Let $\mathcal{F} = \mathcal{Y} - f\ (\mathcal{X} - \mathcal{M})$. Then, $\mathcal{F} \in \delta -$ ßC$(\mathcal{Y},\ \mathcal{T}^*)$. Since $f^{-1}(\mathcal{V}) \subseteq \mathcal{M}$, we have, $f\ (\mathcal{X} - \mathcal{M}) \subseteq (\mathcal{Y} - \mathcal{V})$ so, $\mathcal{V} \subseteq \mathcal{F}$. Also$f^{-1}(\mathcal{F}) = \mathcal{X} - f^{-1}[f(\mathcal{X} - \mathcal{M})] \subseteq \mathcal{X} - (\mathcal{X} - \mathcal{M}) = \mathcal{M}$.

Theorem 3.4: A mapping$f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$is $\delta - ß -$ open if and only if $f[Int(\mathcal{A})] \subseteq \delta - ß - Int[f(\mathcal{A})], for\ every\ \mathcal{A} \subseteq \mathcal{X}$.

Proof: $(\Longrightarrow)$Supposethat $\mathcal{A} \subseteq \mathcal{X}$ and$x \in$ Int$(\mathcal{A})$.

Then there exists an open set $\mathcal{U}_x$ in $\mathcal{X}$ such that $x \in \mathcal{U}_x \subseteq \mathcal{A}$.

Now $f\ (x) \in f\ (\mathcal{U}_x) \subseteq f(\mathcal{A})$, Since f $is\ \delta - ß -$ open, $f\ (\mathcal{U}_x) \in \delta -$ ßΣ$(\mathcal{Y},\ \mathcal{T}^*)$.

Then, $f(x) \in \delta - ß - Int[\ f\ (\mathcal{A})]$. Hence, $f[Int(\mathcal{A})] \subseteq \delta - ß - Int[f(\mathcal{A})]$.

(**Conversely**), Let $\mathcal{U}$ be an open set in $\mathcal{X}$. Then by assumption, $f[Int(\mathcal{U})] \subseteq \delta - ß - Int[f(\mathcal{U})]$. Since $\delta - ß - Int[\ f\ (\mathcal{U})] \subseteq f\ (\mathcal{U})$, and $f\ (\mathcal{U}) = \delta - ß - Int[\ f(\mathcal{U})]$. Therefore$f\ (\mathcal{U}) \in \delta -$ ßΣ$(\mathcal{Y},\ \mathcal{T}^*)$, So f is $\delta - ß -$ open.

Remark 3.5: The equality in the **Theorem**(**3.4**) need not be true as shown in the following example.

Example 3.6: Let $\mathcal{X} = \mathcal{Y} = \{x, y\}$, and $\mathcal{T}$ be the indiscrete topology defined on $\mathcal{X}$ and $\mathcal{T}^*$ be the discrete topology defined on $\mathcal{Y}$ as follows $\mathcal{T}^* = \{\varphi,\ \mathcal{Y}, \{x\}, \{y\}\}$. Then we have $\delta -$ ßΣ$(\mathcal{X},\ \mathcal{T}) = \{\varphi,\ \mathcal{X}, \{x\}, \{y\}\}$ and $\delta -$ ßΣ$(\mathcal{Y},\ \mathcal{T}^*) = \mathcal{T}^*$, Let $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ be the identity function and $\mathcal{A} = \{x\}$. Then, $f[Int(\mathcal{A})] = \varphi$, and $\delta - ß - Int[f(\mathcal{A})] = \{x\}$.

Theorem 3.7: A mapping$f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is$\delta - ß -$ open if and only if

7. $Int[f^{-1}(\mathcal{B})] \subseteq f^{-1}[\delta - ß - Int(\mathcal{B})], \forall\ \mathcal{B} \subseteq \mathcal{Y}$.

Proof: $(\Longrightarrow)$ Let $\mathcal{B}$ be any subset of $\mathcal{Y}$. Then$f$ [Int$(\ f^{-1}(\mathcal{B}))] \subseteq f\ [\ f^{-1}(\mathcal{B})] \subseteq \mathcal{B}$. But

8. f [Int$(\ f^{-1}(\mathcal{B}))] \in \delta -$ ßΣ$(\mathcal{Y},\ \mathcal{T}^*)$, since $Int(\ f^{-1}(\mathcal{B}))$ is open in $\mathcal{X}$ and f is $\delta - ß -$ open.

9. Hence, $f\ [Int(f^{-1}(\mathcal{B}))] \subseteq \delta - ß - Int(\mathcal{B})$. Therefore, $Int[f^{-1}(\mathcal{B})] \subseteq f^{-1}[\delta - ß - Int(\mathcal{B})]$.

10. **(Conversely):** Assume that $\mathcal{A}$ is any subset of $\mathcal{X}$. Then $f\ (\mathcal{A}) \subseteq \mathcal{Y}$. Thus via assumption, we

11. have, $Int(\mathcal{A}) \subseteq Int[f^{-1}(\ f(\mathcal{A})] \subseteq\ f^{-1}[\delta - ß - Int(f\ (\mathcal{A}))]$. Thus,

$f\,[Int(\mathcal{A})] \subseteq \delta - ß - Int[\,f\,(\mathcal{A})], \forall\, \mathcal{A} \subseteq \mathcal{X}$. Hence, via **Theorem (3.4),** f is $\delta - ß -$ open.

Theorem 3.8: A mapping $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is $\delta - ß -$ open if and only if

$f^{-1}[\delta - ß - \mathrm{Cl}(\mathcal{B})] \subseteq \mathrm{Cl}[f^{-1}[(\mathcal{B})]]$, for every $\mathcal{B} \subseteq \mathcal{Y}$.

Proof: Suppose that f is $\delta - ß -$ open and $\mathcal{B} \subseteq \mathcal{Y}$, and let $x \in f^{-1}[\delta - ß - \mathrm{Cl}(\mathcal{B})]$. Then, $f\,(x) \in \delta - ß - \mathrm{Cl}(\mathcal{B})$. Let $\mathcal{U}$ be an open subset in $\mathcal{X}$ such that $x \in \mathcal{U}$.

Since f is $\delta - ß -$ open, then $f\,(\mathcal{U}) \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$ .Therefore $\mathcal{B} \cap f(\mathcal{U}) \neq \varphi$. Then,

$\mathcal{U} \cap f^{-1}(\mathcal{B}\,) \neq \varphi$. Hence $x \in \mathrm{Cl}[\,f^{-1}(\mathcal{B}\,)]$. So, we get $f^{-1}[\delta - ß - \mathrm{Cl}(\mathcal{B})] \subseteq \mathrm{Cl}[f^{-1}[(\mathcal{B})]$.

(**Conversely**): Let $\mathcal{B}$ be subset of $\mathcal{Y}$. Then $(\mathcal{Y} - \mathcal{B}) \subseteq \mathcal{Y}$. via hypothesis, we have,

$f^{-1}[\delta - ß - \mathrm{Cl}(\mathcal{Y} - \mathcal{B})] \subseteq \mathrm{Cl}[\,f^{-1}(\mathcal{Y} - \mathcal{B})]$. This implies,

$\mathcal{X} - \mathrm{Cl}[f^{-1}(\mathcal{Y} - \mathcal{B})] \subseteq \mathcal{X} - f^{-1}[\delta - ß - \mathrm{Cl}(\mathcal{Y} - \mathcal{B})]$. Hence

$\mathcal{X} - \mathrm{Cl}[\mathcal{X} - f^{-1}(\mathcal{B})] \subseteq f^{-1}[\,(\mathcal{Y} - \delta - ß - \mathrm{Cl}(\mathcal{Y} - \mathcal{B}))]$.

Now $\mathcal{X} - \mathrm{Cl}[\mathcal{X} - f^{-1}(\mathcal{B})] = Int[\mathcal{X} - (\mathcal{X} - f^{-1}(\mathcal{B})] = Int[f^{-1}(\mathcal{B})]$. Then, we have

$\mathrm{Y} - \delta - ß - \mathrm{Cl}(\mathcal{Y} - \mathcal{B}) = \delta - ß - Int[\mathcal{Y} - (\mathcal{Y} - \mathcal{B})] = \delta - ß - \mathrm{I}nt(\mathcal{B})$. Therefore,

$Int[\,f^{-1}(\mathcal{B})] \subseteq f^{-1}[\delta - ß - \mathrm{I}nt(\mathcal{B})]$. So, via **Theorem(3.7)** we get f is $\delta - ß -$ open.

Definition 3.9: A mapping $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is said to be a:

Open [**17**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is open of $(\mathcal{Y},\ \mathcal{T}^*)$.

$\alpha -$ open [**18**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is α-open of $(\mathcal{Y},\ \mathcal{T}^*)$.

Semi open [**19**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is semi open of $(\mathcal{Y},\ \mathcal{T}^*)$.

Pre $-$ open [**14**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is Pre-open of $(\mathcal{Y},\ \mathcal{T}^*)$.

$b -$ open[**4**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is b-open of $(\mathcal{Y},\ \mathcal{T}^*)$.

ß $-$ open [**15**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is *ß*-open of $(\mathcal{Y},\ \mathcal{T}^*)$.

E $-$ open [**7**], if the image of each open subset of $(\mathcal{X},\ \mathcal{T})$ is E-open of $(\mathcal{Y},\ \mathcal{T}^*)$.

Quasi $E -$ open [**7**], if the image of every E-open set in $(\mathcal{X},\ \mathcal{T})$ is open in $(\mathcal{Y},\ \mathcal{T}^*)$.

strongly $E-$ open [**7**], if the image of every E-open set in $(\mathcal{X},\ \mathcal{T})$ is E-open in$(\mathcal{Y},\ \mathcal{T}^{*})$.

Remark 3.10: **"**The relationships between $\delta-$ ß $-$ open mappings and other corresponding forms of generalized mappings are shown in the following figure".

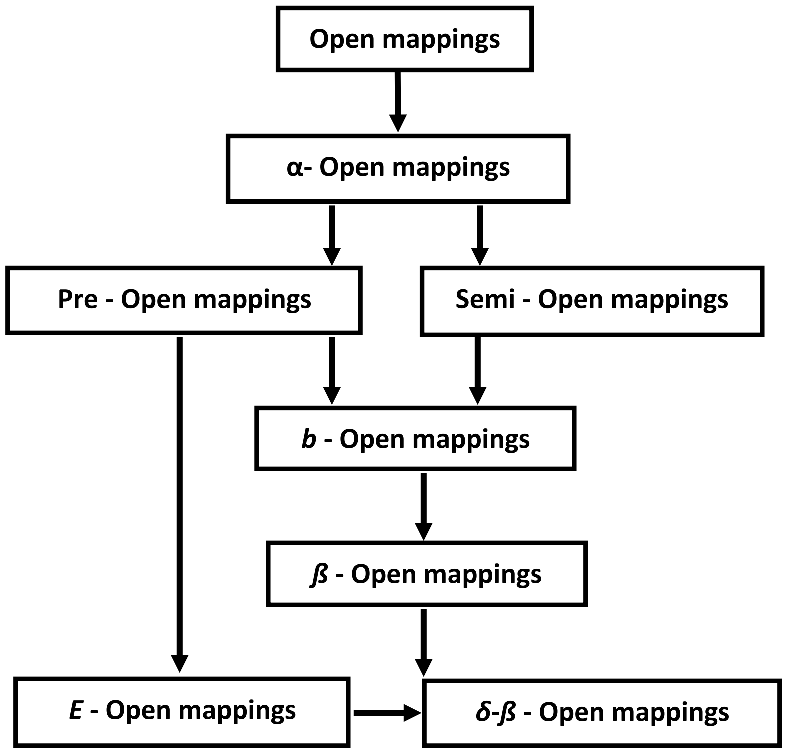

Figure (2): the relationships between $\boldsymbol{\delta}-$ ß $-$ open mappings and other of well-known types of generalized open mappings

However, none of these implications is reversible as shown via the example **(2.3)** of [**20**], and the following example:

Examples 3.11: Let $\mathcal{X}=\ \mathcal{Y}=\{a,b,c,d\}$, define a topology $T=\{\varphi,\mathcal{X},\{a\},\{c\},\{a,b\},\{a,c\},\{a,b,c\},\{a,c,d\}\}$ and define a mapping $f\colon(\mathcal{X},\ \mathcal{T})\longrightarrow(\mathcal{Y},\ \mathcal{T}^{*})$ as follows: $f(a)=a, f(b)=d, f(c)=b$ and $f(d)=c$. Then clearly that, f is$\delta-$ ß $-$ open, but it is not $E-$ open mapping. As well as, $f$neither ß $-$ open nor $b-$ open mapping.

ChARACTERIZATIONS OF QUASI $\boldsymbol{\delta}-$ ß $-$ OPEN MAPPINGS

In this part, we obtain some characterizations and several properties concerning of quasi $\delta-$ ß $-$ open mappings by using $\delta-$ ß $-$ open.

Definition 4.1: A mapping $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is called quasi $\delta -$ ß $-$ open if $f\ (\mathcal{U})$ is open in $\mathcal{Y}$ for every $\mathcal{U} \in \delta -$ ß$\Sigma(\mathcal{X}, \mathcal{T})$.

Remark 4.2: (**a**) $-$ It is obvious that, the concepts quasi of $\delta -$ ß $-$ openness and $\delta -$ ß $-$ continuity are identical if the mapping f is a bijection.

Remark 4.3: From the definitions of (**3.9**) and (**4.1**), it is clear that every quasi $\delta -$ ß $-$ open mapping is open as well as $\delta -$ ß $-$ open, and we have the following diagram:

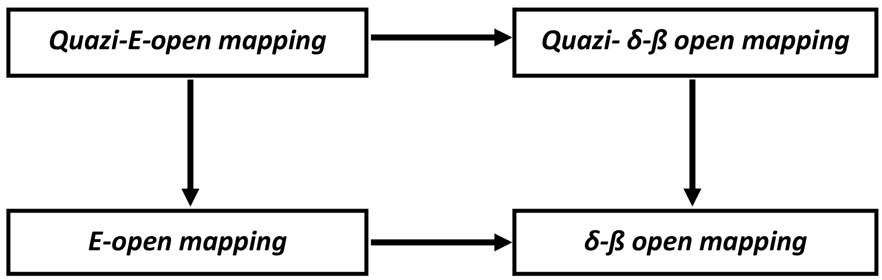

Figure (3): the relationships between quasi $\delta -$ ß $-$ open mappings and other of well-known types of generalized open mappings

"However, the converses of the implications are not true in general as shown in the following example".

Examples 4.4: 1 $-$ Let $\mathcal{X} = \mathcal{Y} = \{1, 2, 3\}$, define a topology $\mathcal{T} = \{\varphi, \mathcal{X}, \{1\}, \{2, 3\}\}$.

Defined a mapping $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$: $f(1) = 1$, $f(2) = 2$ *and* $f(3) = 3$.

Then obviously that f is open. Also, $E -$ open and $\delta -$ ß $-$ open, but it is neither quasi $\delta -$ ß $-$ open nor quasi $E -$ open.

Theorem 4.5: A mapping $f: (\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is quasi $\delta -$ ß $-$ open if and only if for every subset $\mathcal{B}$ *of* $\mathcal{X}$, $f\ (\delta -$ ß $- Int\ (\mathcal{B})) \subseteq Int(f\ (\mathcal{B}))$.

Proof: suppose that f *is* quasi $\delta -$ ß $-$ open mapping. Now, we have $Int\ (\mathcal{B}) \subseteq \mathcal{B}$ and $\delta -$ ß $- Int\ (\mathcal{B})$ *is a* $\delta -$ ß $-$ open set. Therefore, we get that $f\left(\delta -\text{ß} - Int\ (\mathcal{B})\right) \subseteq f\ (\mathcal{B})$.

Since $f\left(\delta -\text{ß} - Int\ (\mathcal{B})\right)$ *is* open, so $f\ (\delta -$ ß $- Int\ (\mathcal{B})) \subseteq Int(f\ (\mathcal{B}))$.

(**Conversely**), suppose that $\mathcal{B}$ is a $\delta -$ ß $-$ open set of $\mathcal{X}$. So, $f\ (\mathcal{B}) = f\left(\delta -\text{ß} - Int\ (\mathcal{B})\right) \subseteq Int\left(f\ (\mathcal{B})\right)$. But, $Int\left(f\ (\mathcal{B})\right) \subseteq f\ (\mathcal{B})$.

Consequently, $Int\left(f\ (\mathcal{B})\right) = f\ (\mathcal{B})$ *and* thus, f is quasi $\delta -$ ß $-$ open mapping.

Definition 4.6: A subset $\mathcal{U}$ of a space $\mathcal{X}$ is said to be $\delta-ß-$ neighborhood of a point $x \in \mathcal{X}$ if there exists an $\delta-ß-$ open set $\mathcal{A}$ of $\mathcal{X}$ such that $x \in \mathcal{A} \subseteq \mathcal{U}$.

Theorem 4.7: For mapping $f:(\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ the following properties are equivalent:

f is quasi $\delta-ß-$ open;

For each subset $\mathcal{A}$ of a space $\mathcal{X}$, $f\ (\delta-ß-Int(\mathcal{A})) \subseteq Int(\ f(\mathcal{A}))$;

$\forall\ x \in \mathcal{X}$ and $\forall\ \delta-ß-$ neighborhood $\mathcal{U}$ of $x \in \mathcal{X}$, there exists a neighborhood $\mathcal{V}$ of $f\ (x)$ in $\mathcal{Y}$ such that $\mathcal{V} \subseteq f\ (\mathcal{U})$.

Proof: $(\mathbf{a}) \Rightarrow (\mathbf{b})$. It follows from Theorem $(\mathbf{4.5})$.

$(\mathbf{b}) \Rightarrow (\mathbf{c})$. Suppose that, $x \in \mathcal{X}$ and $\mathcal{U}$ is an arbitrary $\delta-ß-$ neighborhood of x. Then $\exists$ $\mathcal{V} \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$ (s.t) $x \in \mathcal{V} \subseteq \mathcal{U}$. Then via part $(\mathbf{b})$, we obtain, $f\ (\mathcal{V}) = f\ [\delta-ß-\mathrm{Int}(\mathcal{V})] \subseteq \mathrm{Int}(f(\mathcal{V}))$ and hence, $f\ (\mathcal{V}) = \mathrm{Int}(f(\mathcal{V})$. Therefore, it is follow that, $f\ (\mathcal{V})$ is open in $\mathcal{Y}$ such that $f\ (x) \in f\ (\mathcal{V}) \subseteq f\ (\mathcal{U})$.

$(\mathbf{c}) \Rightarrow (\mathbf{a})$. Suppose that $\mathcal{U} \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$. Then for each $y \in f\ (\mathcal{U})$, there exists a neighborhood $\mathcal{V}_y$ of y in $\mathcal{Y}$ such that, $\mathcal{V}_y \subseteq f\ (\mathcal{U})$. Since $\mathcal{V}_y$ is a neighborhood of y, there exists an open set $\mathcal{W}_y$ in $\mathcal{Y}$ such that $y \in \mathcal{W}_y \subseteq \mathcal{V}_y$. So, $f\ (\mathcal{U}) = \cup\{\mathcal{W}_y : y \in f\ (\mathcal{U})\}$ which is an open set in $\mathcal{Y}$. This implies that f is quasi $\delta-ß-$ open mapping.

Theorem 4.8: A mapping $f:(\mathcal{X},\ \mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is quasi $\delta-ß-$ open if and only if for every subset $\mathcal{B}$ of $\mathcal{Y}$, $\delta-ß-Int\ (f^{-1}(\mathcal{B})) \subseteq f^{-1}(Int(\mathcal{B}))$.

Proof: Let $\mathcal{B}$ be any subset of $\mathcal{Y}$. Then, $\delta-ß-Int\left(f^{-1}(\mathcal{B})\right) \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$ and f is quasi $\delta-ß-$ open, then $f\ (\delta-ß-Int(\ f^{-1}(\mathcal{B})))$

$$\subseteq \mathrm{Int}(f\ (\ f^{-1}(Int(\mathcal{B})))) \subseteq$$

$Int(\mathcal{B})$. Thus, $\delta-ß-Int\ (f^{-1}(\mathcal{B})) \subseteq f^{-1}(Int(\mathcal{B}))$.

(**Conversely**), Let $\mathcal{U} \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$. Then via hypothesis $\delta-ß-Int[\ f^{-1}(\ f\ (\mathcal{U}))] \subseteq f^{-1}[Int(f\ (\mathcal{U}))]$. Then, $\delta-ß-Int(\mathcal{U}) \subseteq f^{-1}[Int(\ f\ (\mathcal{U}))]$,

but, $\delta-ß-Int(\mathcal{U}) = \mathcal{U}$. Hence, $\mathcal{U} \subseteq f^{-1}\left[Int\left(f\ (\mathcal{U})\right)\right]$ & $f(\mathcal{U})$

$$\subseteq Int\left(f\ (\mathcal{U})\right).$$

Consequently , f is quasi $\delta - ß -$ open mapping.

Theorem 4.9: A mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is quasi $\delta - ß -$ open if and only if for every subset $\mathcal{B}$ of $\mathcal{Y}$, and for every set $\mathcal{M} \in \delta - ßC(\mathcal{X}, \mathcal{T})$ containing $f^{-1}(\mathcal{B})$, there exists a closed subset $\mathcal{F}$ of $\mathcal{Y}$ containing $\mathcal{B}$ such that, $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$.

Proof: suppose that, f *is* quasi $\delta - ß -$ open and $\mathcal{B} \subseteq \mathcal{Y}$, and let $\mathcal{M} \in \delta - ßC(\mathcal{X}, \mathcal{T})$ with $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$. Now, put $\mathcal{F} = \mathcal{Y} - f(\mathcal{X} - \mathcal{M})$. It is obvious that $f^{-1}(\mathcal{B}) \subseteq \mathcal{M} \Longrightarrow \mathcal{B} \subseteq \mathcal{F}$.

Since, f is quasi $\delta - ß - $ open mapping, we get $\mathcal{F}$ as a closed subset of $\mathcal{Y}$.

Also, we obtain $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$.

(**Conversely**), Let $\mathcal{U} \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$ and put $\mathcal{B} = \mathcal{Y} - f(\mathcal{U})$. Then, $\mathcal{X} - \mathcal{U} \in \delta - ßC(\mathcal{X}, \mathcal{T})$ with $f^{-1}(\mathcal{B}) \subseteq \mathcal{X} - \mathcal{U}$. Via assumption, there exists a closed set $\mathcal{F}$ of $\mathcal{Y}$ such that: $\mathcal{B} \subseteq \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subseteq \mathcal{X} - \mathcal{U}$. Hence, we get $f(\mathcal{U}) \subseteq \mathcal{Y} - \mathcal{F}$. On the other hand, it follows that $\mathcal{B} \subseteq \mathcal{F}, \mathcal{Y} - \mathcal{F} \subseteq \mathcal{Y} - \mathcal{B} = f(\mathcal{U})$. Thus, we have $f(\mathcal{U}) = \mathcal{Y} - \mathcal{F}$ which is open and hence, f is quasi $\delta - ß -$ open mapping.

Theorem 4.10: A mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is quasi $\delta - ß -$ open if and only if for every subset $\mathcal{B}$ *of* $\mathcal{Y}$, $f^{-1}(Cl(\mathcal{B})) \subseteq \delta - ß - Cl(f^{-1}(\mathcal{B}))$.

Proof: Assume that, f *is* quasi $\delta - ß -$ open mapping. For any subset $\mathcal{B}$ *of* $\mathcal{Y}$, $f^{-1}(\mathcal{B}) \subseteq \delta - ß - \mathrm{Cl}[f^{-1}(\mathcal{B})]$. Therefore by Theorem(**4.9**), there exists a closed set $\mathcal{F}$ of $\mathcal{Y}$ such that $\mathcal{B} \subseteq \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subseteq \delta - ß - \mathrm{Cl}(f^{-1}(\mathcal{B}))$. Therefore, we obtain, $f^{-1}(Cl(\mathcal{B})) \subseteq f^{-1}(\mathcal{F}) \subseteq \delta - ß - Cl(f^{-1}(\mathcal{B}))$.

(**Conversely**), Suppose that $\mathcal{B}$ is any subset of a space $\mathcal{Y}$, and $\mathcal{M} \in \delta - ßC(\mathcal{X}, \mathcal{T})$ with $f^{-1}(\mathcal{B}) \subseteq \mathcal{M}$. Put $\mathcal{F} = Cl(\mathcal{B})$, then we have $\mathcal{B} \subseteq \mathcal{F}$ and $\mathcal{F}$ is closed and $f^{-1}(\mathcal{F}) \subseteq \delta - ß - Cl(f^{-1}(\mathcal{B})) \subseteq \mathcal{M}$. Then by Theorem **(4.9),** we get f isquasi $\delta - ß -$ open

Theorem 4.11: Let $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ and $g: (\mathcal{Y}, \mathcal{T}^*) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ be two mappigs and $gof: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ is quasi $\delta -$ ß $-$ open map. If g is continuous injective, then f is quasi $\delta -$ ß $-$ open mapping.

Proof: Let $\mathcal{U}$ be $\delta -$ ß $-$ open set in $\mathcal{X}$, then $(gof)(\mathcal{U})$ is open in $\mathcal{Z}$ since gof is quasi $\delta -$ ß $-$ open. Again g is an injective continuous mapping, $f(\mathcal{U}) = g^{-1}(gof(\mathcal{U}))$ is open in $\mathcal{Y}$. This shows that f is quasi $\delta -$ ß $-$ open mapping.

Definition 4.12: A space $(\mathcal{X}, \mathcal{T})$ is said to be:

$\delta -$ ß $- T_1$ [**21**], if for each pair of distinct points x and y of $\mathcal{X}$, there exist $\delta -$ ß $-$ open sets $\mathcal{A}$ and $\mathcal{B}$ containing x and y, respectively, such that, $x \notin \mathcal{B}$ and $y \notin \mathcal{A}$.

$\delta -$ ß $- T_2$ [**21**], if for each pair of distinct points x and y of $\mathcal{X}$ there exist disjoint $\delta -$ ß $-$ open sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{X}$ such that $x \in \mathcal{A}$ *and* $y \in \mathcal{B}$.

Theorem 4.13: The following properties arehold for quasi $\delta -$ ß $-$ open bijective mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$:

If $\mathcal{X}$ is $\delta -$ ß $- T_1$ then $\mathcal{Y}$ is T_1.

If $\mathcal{X}$ is $\delta -$ ß $- T_2$ then $\mathcal{Y}$ is T_2.

Proof: (**a**) $-$ Let y_1 and y_2 be any two distinct points in $\mathcal{Y}$. Then there exist x_1 and x_2 in $\mathcal{X}$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $\mathcal{X}$ is $\delta -$ ß $- T_1$ then, there exist two $\delta -$ ß $-$ open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{X}$ with $x_1 \in \mathcal{U}, x_2 \notin \mathcal{U}$ and $x_2 \in \mathcal{V}, x_1 \notin \mathcal{V}$. Now $f(\mathcal{U})$ and $f(\mathcal{V})$ are open in $\mathcal{Y}$ with $y_1 \in f(\mathcal{U}), y_2 \notin f(\mathcal{U})$ and $y_2 \in f(\mathcal{V}), y_1 \notin f(\mathcal{V})$.

Proof: **(b) -** is similar to **(a).** Thus is omitted.

Definition 4.14: A space $(\mathcal{X}, \mathcal{T})$ is said to be:

$\delta -$ ß $-$ compact [**22**], if every cover of $\mathcal{X}$ by $\delta -$ ß $-$ open sets has a finite sub cover.

$\delta -$ ß $-$ Lindelof if every cover of $\mathcal{X}$ by $\delta -$ ß $-$ open sets has a countable subcover.

Theorem 4.15: The following properties are hold for quasi $\delta -$ ß $-$ open bijective mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$:

If $\mathcal{Y}$ is compact, then $\mathcal{X}$ is $\delta -$ ß $-$ compact

If $\mathcal{Y}$ is Lindelof, then $\mathcal{X}$ is $\delta -$ ß $-$ Lindelof.

Proof: **(a)** $-$ Let $\mathcal{D}_1 = \{\mathcal{U}_\lambda : \lambda \in \Delta\}$ be an $\delta -$ ß $-$ open cover of $\mathcal{X}$. Then $\mathcal{K}_1 = \{f(\mathcal{U}_\lambda) : \lambda \in \Delta\}$ is a cover of $\mathcal{Y}$ via open sets in $\mathcal{Y}$. Since $\mathcal{Y}$ is compact, Then $\mathcal{K}_1$ has a finite subcover

$\mathcal{K}_2 = \{f(\mathcal{U}_{\lambda_1}), f(\mathcal{U}_{\lambda_2}), \ldots\ldots.., f(\mathcal{U}_{\lambda_n})\}$ for $\mathcal{Y}$. Then $\mathcal{D}_2 = \{\mathcal{U}_{\lambda_1}, \mathcal{U}_{\lambda_2}, \ldots\ldots, \mathcal{U}_{\lambda_n}\}$ is a finite subcover of $\mathcal{D}_1$ for $\mathcal{X}$. ***Proof***: (**b**) − is similar to(**a**). Thus is omitted.

Definition 4.16: A space $(\mathcal{X}, \mathcal{T})$ is said to be $\delta -$ ß $-$ Connected [**21**] if $\mathcal{X}$ cannot be written as the union of two nonempty disjoint $\delta -$ ß $-$ open sets.

Theorem 4.17: If $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is quasi $\delta -$ ß $-$ open surjective mapping and $\mathcal{Y}$ is Connected. Then $\mathcal{X}$ is $\delta -$ ß $-$ Connected.

Proof: Assume that $\mathcal{X}$ is not $\delta -$ ß $-$ Connected. Then there exist two non − empty disjoint $\delta -$ ß $-$ open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{X}$ such that $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$. Then $f(\mathcal{U})$ and $f(\mathcal{V})$ are non − empty disjoint open sets in $\mathcal{Y}$ with $\mathcal{Y} = f(\mathcal{U}) \cup f(\mathcal{V})$ which contradicts the fact that $\mathcal{Y}$ is connected.

CHARACTERIZATIONS OF STRONGLY $\delta -$ ß $-$ OPEN MAPPINGS

In this section, we obtain some characterizations and several properties concerning of strongly $\delta -$ ß $-$ open mappings via $\delta -$ ß $-$ open sets.

Definition 5.1: A mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is said to be strongly $\delta -$ ß $-$ open if $f(\mathcal{U}) \in \delta -$ ß$\Sigma(\mathcal{Y}, \mathcal{T}^*)$ for every $\mathcal{U} \in \delta -$ ß$\Sigma(\mathcal{X}, \mathcal{T})$.

Remark 5.2: From the definitions of **(3.9)** and **(5.1),** it is clear that every strongly $\delta -$ ß $-$ open mapping is open as well as $\delta -$ ß $-$ open, and we have the following figure:

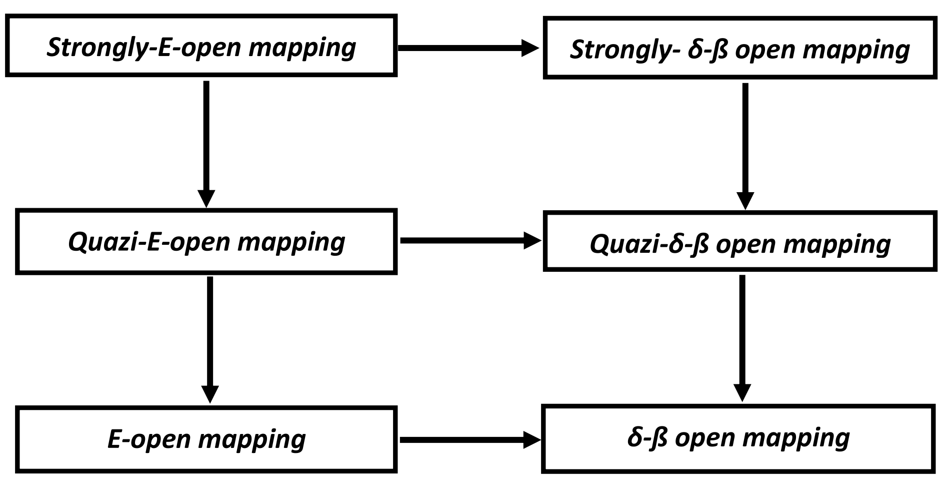

Figure (3): the relationships between strongly $\delta -$ ß $-$ open mappings and other of well-known kinds of generalized open mappings

"However, the converses of the implications are not true in general as shown in the following examples".

Examples 5.3: Let $\mathcal{X} = \mathcal{Y} = \{x, y, w, z\}$, define a topology $T = \{\varphi, \mathcal{X}, \{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, w, z\}\}$ and define a mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ as follows: $f(x) = x, f(y) = z, f(w) = y$ *and* $f(z) = w$. Then clearlythat, f is $\delta -$ ß $-$ open, but it is not $E -$ open mapping. Moreover f neither strongly $\delta -$ ß $-$ open nor strongly $E -$ open.

Examples 5.4: Let $\mathcal{X} = \mathcal{Y} = \{a, b, c, d\}$, define atopology $T = \{\varphi, \mathcal{X}, \{b\}, \{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}, \{a, b, d\}\}$ and define a mapping $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ as follows: $f(a) = a, f(b) = d, f(c) = b$ *and* $f(d) = c$.
Then, f is strongly $\delta -$ ß $-$ open , but not strongly $E -$ open mapping, since there exists the subset $\{a, b\} \in E\Sigma(\mathcal{X}, \mathcal{T})$, but, $f(\{a, b\}) = \{a, d\} \notin E\Sigma(\mathcal{Y}, \mathcal{T}^*)$.

Definition 5.5: A topological space$(\mathcal{X}, \mathcal{T})$ is said to be a $T_{\delta-ß} -$ space [**23**] if every $\delta -$ ß $-$ open subset of $(\mathcal{X}, \mathcal{T})$ is open in $(\mathcal{X}, \mathcal{T})$.

Remark 5.6: If $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is quasi$\delta -$ ß $-$ open mapping and $\mathcal{Y}$ is a $T_{\delta-ß} -$ space, then quasi $\delta -$ ß $-$ openness coincide with strongly $\delta -$ ß $-$ openness.

Definition 5.7: A mapping$f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is said to be $\delta -$ ß $-$ irresolute [**21**]if $f^{-1}(\mathcal{V})$ is $\delta -$ ß $-$ open in $\mathcal{X}$ for every $\delta -$ ß $-$ open set $\mathcal{V}$ of $\mathcal{Y}$.

Theorem 5.8: Suppose that $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ and $g: (\mathcal{Y}, \mathcal{T}^*) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ are two strongly $\delta -$ ß $-$ open mappings. Then, $gof: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ is strongly $\delta -$ ß $-$ open mapping.

Proof: The proof is obvious thus omitted.

Theorem 5.9: A mapping$f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is strongly$\delta -$ ß $-$ open if and only if $\forall\ x \in \mathcal{X}$ and for each $\mathcal{U} \in \delta - ß\Sigma(\mathcal{X},\mathcal{T})$ with $x \in \mathcal{U}$, there exists $\mathcal{V} \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$ such that, $f(x) \in \mathcal{V}$ and $\mathcal{V} \subseteq f\ (\mathcal{U})$.

Proof: It is clear thus deleted.

Theorem 5.10: A mapping $f: (\mathcal{X},\mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is strongly $\delta - ß -$ open if and only if $\forall\ x \in \mathcal{X}$ and $\forall\ \delta - ß -$ neighborhood $\mathcal{U}$ of $x \in \mathcal{X}$, $\exists$ an $\delta - ß -$ neighborhood $\mathcal{V}$ of $f(x)$ in $\mathcal{Y}$ such that, $\mathcal{V} \subseteq f\ (\mathcal{U})$.

Proof: suppose that $\mathcal{U}$ is an $\delta - ß -$ neighborhood of $x \in \mathcal{X}$. Then $\exists\ \mathcal{W} \in \delta - ß\Sigma(\mathcal{X},\mathcal{T})$ $x \in \mathcal{W} \subseteq \mathcal{U}$. So , $f(x) \in f(\mathcal{W}) \subseteq f(\mathcal{U})$. Since f is strongly $\delta - ß -$ open, this implies, $f(\mathcal{W}) \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$. So $\mathcal{V} = f\ (\mathcal{W})$ is an $\delta - ß -$ neighborhoodof $f(x)$ & $V \subseteq f(\mathcal{U})$.

(**Conversely**), Let $\mathcal{U} \in \delta - ß\Sigma(\mathcal{X},\mathcal{T})$ and $x \in \mathcal{U}$, then $\mathcal{U}$ is an $\delta - ß -$ neighborhood of x. So via supposition, there exists an $\delta - ß -$ neighborhood $\mathcal{V}_{f(x)}$ of $f(x)$ such that, $f(x) \in \mathcal{V}_{f(x)} \subseteq f(\mathcal{U})$. It follows $f(\mathcal{U})$ is an $\delta - ß -$ neighborhoodof of each of its points. Therefore, $f(\mathcal{U}) \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$, hence f is strongly $\delta - ß -$ open

Theorem 5.11: A mapping $f: (\mathcal{X},\mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is strongly $\delta - ß -$ open if and only if $f\left(\delta - ß - Int(\mathcal{A})\right) \subseteq \delta - ß - Int\left(f(\mathcal{A})\right)$ for each $\mathcal{A} \subseteq \mathcal{X}$.

Proof: Assume that $\mathcal{A} \subseteq \mathcal{X}$ and $x \in \delta - ß - Int(\mathcal{A})$. So, $\mathcal{U}_x \in \delta - ß\Sigma(\mathcal{X},\mathcal{T})$ such that , $x \in \mathcal{U}_x \subseteq \mathcal{A}$. Then, $f(x) \in f(\mathcal{U}_x) \subseteq f(\mathcal{A})$ and via assumption, $f(\mathcal{U}_x) \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$. Hence, $f(x) \in \delta - ß - Int\left(f(\mathcal{A})\right)$. Therefore, $f\left(\delta - ß - Int(\mathcal{A})\right) \subseteq\ \delta - ß - Int\left(f(\mathcal{A})\right)$.

(**Conversely**), Let $\mathcal{U} \in \delta - ß\Sigma(\mathcal{X},\mathcal{T})$. *So* via assumption, $f\left(\delta - ß - Int(\mathcal{U})\right) \subseteq\ \delta - ß - Int\left(f(\mathcal{U})\right)$. Since $\delta - ß - Int(\mathcal{U}) = \mathcal{U}$ and $\delta - ß - Int\left(f(\mathcal{U})\right) \subseteq f(\mathcal{U})$. Hence, $f(\mathcal{U}) = \delta - ß - Int\left(f(\mathcal{U})\right)$. Thus, $f(\mathcal{U}) \in \delta - ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$.

Theorem 5.12: A mapping $f:(\mathcal{X},\mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is strongly $\delta-ß-$open if and only if $\delta-ß-Int\big(f^{-1}(\mathcal{B})\big) \subseteq f^{-1}\big(\delta-ß-Int(\mathcal{B})\big)$ for each subset $\mathcal{B}$ of $\mathcal{Y}$.

Proof: Let $\mathcal{B}$ be any subset of $\mathcal{Y}$. Since $\delta-ß-Int\big(f^{-1}(\mathcal{B})\big) \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$ and f is strongly $\delta-ß-$open, then $f\ (\delta-ß-Int(f^{-1}(\mathcal{B}))) \in \delta-ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$. Also we have, $f(\delta-ß-Int(f^{-1}(\mathcal{B}))) \subseteq f\big(f^{-1}(\mathcal{B})\big) \subseteq \mathcal{B}$. So $f(\delta-ß-Int(f^{-1}(\mathcal{B}))) \subseteq \delta-ß-Int\big((\mathcal{B})\big)$

Therefore, $\delta-ß-Int\big(f^{-1}(\mathcal{B})\big) \subseteq f^{-1}\big(\delta-ß-Int(\mathcal{B})\big)$.

(**Conversely**), Assume that $\mathcal{A} \subseteq \mathcal{X}$. Then $f(\mathcal{A}) \subseteq \mathcal{Y}$. Hence via assumption, we obtain, $\delta-ß-Int\big(f(\mathcal{A})\big) \subseteq \delta-ß-Int\big(f^{-1}(f(\mathcal{A}))\big) \subseteq f^{-1}(\delta-ß-Int(f(\mathcal{A})))$. This implies , $f\ \big(\delta-ß-Int(\mathcal{A})\big) \subseteq f(f^{-1}(\delta-ß-Int(f(\mathcal{A})))) \subseteq \delta-ß-Int\big(f(\mathcal{A})\big)$. Thus, $f\ \big(\delta-ß-Int(\mathcal{A})\big) \subseteq \delta-ß-Int\big(f(\mathcal{A})\big)\ \forall\ \mathcal{A} \subseteq \mathcal{X}$. Thus, by **Theorem** (**5.11**), we get f is strongly $\delta-ß-$open.

Theorem 5.13: A mapping $f:(\mathcal{X},\mathcal{T}) \longrightarrow (\mathcal{Y},\ \mathcal{T}^*)$ is strongly $\delta-ß-$open if and only if $f^{-1}\big(\delta-ß-Cl(\mathcal{B})\big) \subseteq \delta-ß-Cl\big(f^{-1}(\mathcal{B})\big)$ for each subset $\mathcal{B}$ of $\mathcal{Y}$.

Proof: Suppose that, $\mathcal{B} \subseteq \mathcal{Y}$ and $x \in f^{-1}\big(\delta-ß-Cl(\mathcal{B})\big)$. Then $f(x) \in \delta-ß-Cl(\mathcal{B})$.

Let $\mathcal{U} \in \delta-ß\Sigma(\mathcal{X},\mathcal{T})$ (s.t) $x \in \mathcal{U}$, via supposition, $f(\mathcal{U}) \in \delta-ß\Sigma(\mathcal{Y},\ \mathcal{T}^*)$ & $f(x) \in f(\mathcal{U})$,

Hence, $f(\mathcal{U}) \cap \mathcal{B} \neq \varphi$. Thus, $\mathcal{U} \cap f^{-1}(\mathcal{B}) \neq \varphi$. So, $x \in \delta-ß-Cl\big(f^{-1}(\mathcal{B})\big)$. Therefor, we get $f^{-1}\big(\delta-ß-Cl(\mathcal{B})\big) \subseteq \delta-ß-Cl\big(f^{-1}(\mathcal{B})\big)$.

(**Conversely**), Let $\mathcal{B}$ be any subset of $\mathcal{Y}$, then $\mathcal{Y}-\mathcal{B} \subseteq \mathcal{Y}$, via supposition, we have $f^{-1}\big(\delta-ß-Cl(\mathcal{Y}-\mathcal{B})\big) \subseteq \delta-ß-Cl\big(f^{-1}(\mathcal{Y}-\mathcal{B})\big)$. This implies that, $\mathcal{X}-\delta-ß-Cl\big(f^{-1}(\mathcal{Y}-\mathcal{B})\big) \subseteq \mathcal{X}-f^{-1}\big(\delta-ß-Cl(\mathcal{Y}-\mathcal{B})\big)$. Thus, $\mathcal{X}-\delta-ß-Cl\big(\mathcal{X}-f^{-1}(\mathcal{B})\big) \subseteq f^{-1}\big(\mathcal{Y}-\delta-ß-Cl(\mathcal{Y}-\mathcal{B})\big)$. Thus, $\delta-ß-Int\big(f^{-1}(\mathcal{B})\big) \subseteq f^{-1}\big(\delta-ß-Int(\mathcal{B})\big)$. So, via **Theorem** (**5.12**), we obtain f is strongly $\delta-ß-$open.

$\mathbf{Theorem\ 5.14}$: Let $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ be a mapping and $g: (\mathcal{Y}, \mathcal{T}^*) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ be astrongly $\delta - ß -$ open injective. If $gof: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ is $\delta - ß -$ irresolute, then f is $\delta - ß -$ irresolute.

$\boldsymbol{Proof}$: Let $\mathcal{U} \in \delta - ß\Sigma(\mathcal{Y}, \mathcal{T}^*)$. Since g is strongly $\delta - ß -$ open. Then, $g(\mathcal{U}) \in \delta - ß - \Sigma(\mathcal{Z}, \mathcal{T}^{**})$. Aswell as, gof is $\delta - ß -$ irresolute, therefore we obtain $(g\text{o}f)^{-\mathbf{1}}(g(\mathcal{U})) \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$. Since g is injective, so we get
$(g\text{o}f)^{-1}(g(\mathcal{U})) = (f^{-1}\text{o}\ g^{-1})(g(\mathcal{U})) = f^{-1}\left(g^{-1}(g(\mathcal{U}))\right) = f^{-1}(\mathcal{U})$
$\Longrightarrow f^{-1}(\mathcal{U}) \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$.
Then, f is $\delta - ß -$ irresolute.

$\mathbf{Theorem\ 5.15}$: Let $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ be a strongly $\delta - ß -$ open surjective, and $g: (\mathcal{Y}, \mathcal{T}^*) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ be any mapping. If $gof: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ is $\delta - ß -$ irresolute, then g is $\delta - ß -$ irresolute.

$\boldsymbol{Proof}$: Assume that $\mathcal{V} \in \delta - ß\Sigma(\mathcal{Z}, \mathcal{T}^{**})$. Since $g\text{o}f$ is $\delta - ß -$ irresolute, then $(g\text{o}f)^{-\mathbf{1}}(\mathcal{V}) \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$. As well f is strongly $\delta - ß -$ open, so we get $f((g\text{o}f)^{-1}(\mathcal{V})) \in \delta - ß\Sigma(\mathcal{Y}, \mathcal{T}^*)$. Since f is an surjective, Then,
$f[(g\text{o}f)^{-1}(\mathcal{V})] = (f\text{o}(g\text{o}f)^{-1})(\mathcal{V}) = [f\text{o}(f^{-1}\text{o}\ g^{-1})](\mathcal{V})$
$= [(f\text{o}f^{-1})\ \text{o}\ g^{-1}](\mathcal{V}) = g^{-1}(\mathcal{V})$.
Hence, g is $\delta - ß -$ irresolute.

$\mathbf{Theorem\ 5.16}$: Let $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ and $g: (\mathcal{Y}, \mathcal{T}^*) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ be two mappigs such that $gof: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Z}, \mathcal{T}^{**})$ is a strongly $\delta - ß -$ open mappig;
If f is $\delta - ß -$ irresolute surjective, then g is strongly $\delta - ß -$ open;
If g is $\delta - ß -$ irresolute injective, then f is strongly $\delta - ß -$ open.

Proof: **(a)** – Suppose that, $\mathcal{U} \in \delta - ß\Sigma(\mathcal{Y}, \mathcal{T}^*)$. Since f is $\delta - ß -$ irresolute, consequently, $f^{-1}(\mathcal{U}) \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$. Now since gof is strongly $\delta - ß -$ open and f is surjective, then
$g\text{o}f(\ f^{-1}(\mathcal{U})) = \ g(\mathcal{U})$,
$\in \delta - ß\Sigma(\mathcal{Z}, \mathcal{T}^{**})$. This implies that g is strongly $\delta - ß -$ open.

(b)- Assume that, $\mathcal{V} \in \delta - ß\Sigma(\mathcal{X}, \mathcal{T})$. Since gof is strongly $\delta - ß -$ open, therefore $(gof)(\mathcal{V}) \in \delta - ß\Sigma(\mathcal{Z}, \mathcal{T}^{**})$. Again we have, g is $\delta - ß$ $-$ irresolute and injective, thus $g^{-1}(\, gof(\mathcal{V})) = f\,(\mathcal{V}) \in \delta -$ $ß\Sigma(\mathcal{Y}, \mathcal{T}^*)$. This shows that f is strongly $\delta - ß -$ open.

Theorem 5.17: Let $f: (\mathcal{X}, , \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ be a strongly $\delta - ß -$ open bijective mappig. Thenthefollowing statements hold:

If $\mathcal{X}$ is $\delta - ß - \mathrm{T}_1$ space, then $\mathcal{Y}$ is $\delta - ß - \mathrm{T}_1$.

If $\mathcal{X}$ is $\delta - ß - \mathrm{T}_2$ space, then $\mathcal{Y}$ is $\delta - ß - \mathrm{T}_2$.

Proof: (**a**) $-$ Let y_1 and y_2 be any two distinct points in $\mathcal{Y}$. Then $\exists\, x_1$ and x_2 in $\mathcal{X}$, such that $f\,(x_1) = y_1$ and $f\,(x_2) = y_2$. Since $\mathcal{X}$ is $\delta - ß - \mathrm{T}_1$ then, there exist two $\delta - ß -$ open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{X}$with $x_1 \in \mathcal{U}, x_2 \notin \mathcal{U}$ & $x_2 \in \mathcal{V}, x_1 \notin \mathcal{V}$. Now$f\,(\mathcal{U})$ and $f\,(\mathcal{V})$are $\delta - ß -$ open sets in $\mathcal{Y}$ with $y_1 \in f\,(\mathcal{U}), y_2 \notin f\,(\mathcal{U})$ and $y_2 \in f\,(\mathcal{V}), y_1 \notin f\,(\mathcal{V})$.

Proof: **(b) -** is similar to **(a).** Thus is omitted.

Theorem 5.18: If $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ be a strongly $\delta - ß -$ open bijective mappig. Then the following statementshold:

If $\mathcal{Y}$ is $\delta - ß -$ compact space, then $\mathcal{X}$ is $\delta - ß -$ compact.

If $\mathcal{Y}$ is $\delta - ß -$ Lindelof space, then $\mathcal{X}$ is $\delta - ß -$ Lindelof.

Proof: (**a**) $-$ Let $\mathcal{D}_1 = \{\mathcal{U}_\lambda : \lambda \in \Delta\}$ be an $\delta - ß -$ open cover of $\mathcal{X}$. Then $\mathcal{K}_1 = \{f(\mathcal{U}_\lambda) : \lambda \in \Delta\}$is a cover of $\mathcal{Y}$ by $\delta - ß$ $-$ open sets in $\mathcal{Y}$. Since $\mathcal{Y}$ is$\delta - ß -$ compact space, Then $\mathcal{K}_1$ has a finite subcover $\mathcal{K}_2 = \{f\,(\mathcal{U}_{\lambda_1}), f\,(\mathcal{U}_{\lambda_2}), \ldots \ldots .., f\,(\mathcal{U}_{\lambda_n})\}$ for $\mathcal{Y}$. Then $\mathcal{D}_2 = \{\mathcal{U}_{\lambda_1}, \mathcal{U}_{\lambda_2}, \ldots \ldots, \mathcal{U}_{\lambda_n}\}$is a finite subcover of $\mathcal{D}_1$ for $\mathcal{X}$.

Proof: **(b) -** is similar to **(a).** Thus is omitted.

Theorem 5.19: If $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}^*)$ is a strongly $\delta - ß -$ open surjective mappig and $\mathcal{Y}$ is $\delta - ß -$ Connected then, $\mathcal{X}$ is $\delta - ß -$ Connected

Proof: Assume that $\mathcal{X}$ is not $\delta - ß -$ Connected. . Then there exist two non $-$ empty disjoint $\delta - ß -$ open sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{X}$ such that$\mathcal{X} = \mathcal{U} \cup \mathcal{V}$. Then $f\,(\mathcal{U})$and $f\,(\mathcal{V})$

are non − empty disjoint $\delta - ß -$ open sets in $\mathcal{Y}$ with $\mathcal{Y} = f(\mathcal{U}) \cup f(\mathcal{V})$ which contradicts the fact that $\mathcal{Y}$ is $\delta - ß -$ Connected.

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CONCLUSION

Generalized open sets place a significant role in general Topology and it applications. And many topologists worldwide are focusing their researches on these topics and this mounted to many important and useful results. "Indeed a significant theme in General Topology, Real analysis and many other branches of mathematics. One of the well-known concepts and that expected it will has a wide applying in physics and Topology and their applications is the concept of $\delta - ß -$ open sets." "One can observe the influence of general topological spaces in computer sciences and digital topology **[24, 25],** computational topology for geometric and molecular design **[26],** particle physics, high energy physics, quantum physics and Superstring theory **[27,28,29,30]".** In this paper we introduced and investigated the concepts of new classes of mappings such as $\delta - ß -$ open, quasi δ-$ß$-open, and strongly $\delta - ß -$ open mappings which may have very important applications in quantum particle physics, high energy physics and superstring theory. Additionally, the fuzzy topological version of the concepts and results introduced in this paper are very important, since El-Naschie has shown that the notion of fuzzy topology have very important applications in quantum particle physics especially in related to both string theory and ε^{∞} theory **[31].**

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