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C-Conformal Transformation of Berwald and Cartan Connection of Randers Space

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ABSTRACT

In this article, we consider the Randers metric and we proved that it is a weakly Berwald metric. Further, we show that, the difference tensor corresponding to the Cartan and Berwald connections are invariant under C- conformal transformation.

1. Introduction

The concept of (α,β)-metric $L(\alpha,\beta)$ was introduced in 1972 by Matsumoto. Let $F^n = (M^n, L(\alpha,\beta))$ be an n-dimensional Finsler space with an (α,β)-metric $L(\alpha,\beta)$. The fundamental function $L(\alpha,\beta)$ is a positive homogeneous of degree one in α and β, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form in M^n. In F^n, the Riemannian space $R^n = (M^n, \alpha)$ is called an associated Riemannian space with F^n and the Riemannian connection constructed by α is called the associated Riemannian conncetion with F^n, which is denoted by the Christoffel symbol $\left\{ {i \atop jk} \right\}$ of R^n. In F^n, the difference tensors of the Finsler connection are given by the differences of the h-connection co-efficients of the Finsler connection and the associated Riemannian connection. The fundamental Finsler connection are the Cartan connection $C\Gamma = \left(\Gamma^{*i}_{jk}, \mathrm{G}^i_j, \mathrm{C}^i_{jk}\right)$ and Berwald connection $\beta\Gamma = \left(\mathrm{G}^i_{jk}, \mathrm{G}^i_j, 0\right)$.

Definition 1.1. A spray is called a weakly affine spray if the (hv)-Ricci curvature tensor $G_{kl} = 0$ where $G_{kl} = G^{r}_{rkl}$.

We denoted the difference tensors of $C\Gamma$ and $\beta\Gamma$ by D^{i}_{jk} and $'D^{i}_{jk}$, i.e., $D^{i}_{jk} = \Gamma^{*i}_{jk} - \begin{Bmatrix} i \\ jk \end{Bmatrix}$, $'D^{i}_{jk} = G^{i}_{jk} - \begin{Bmatrix} i \\ jk \end{Bmatrix}$ respectively. It is well known [12] that if the covariant vector b_i is parallel with respect to the Riemannian connection, then $D^{i}_{jk} = 0$ and the space becomes a Berwald space.

In the present paper, we consider the Randers metric and we proved that it is of weakly Berwald metric and also we have shown that the difference tensors corresponding to the Cartan connection and Berwald connection are invariant under C-Conformal transformation.

Difference tensors of Randers metric

In this section, we consider an n-dimensional Finsler space $F^n = (M^n, L(\alpha, \beta))$ with Randers metric $L = \alpha + \beta$. Let l^i be the normalize supporting element $\frac{y^i}{L}$. The fundamental metric tensor $g_{ij}(x, y)$ and its reciprocal tensor $g^{ij}(x, y)$ of the Randers space F^n are given by

$$g_{ij} = \tau a_{ij} + b - ib_i + \left(Y_i b_j + Y_j b_i\right) + -\mu Y_i Y_j \qquad (2.1)$$

and

$$g^{ij} = \left\{a^{ij} - \left(l^i b^j + l^j b^i\right) + (b^2 + \mu) l^i l^j\right\}/\tau,$$

Where, we put

$$Y_j = a_{ij} Y^i, \quad Y^i = \frac{y^i}{\alpha}, \; \mu = b_i Y^j, \quad \tau = \frac{L}{\alpha} = (1 + \mu), \qquad b^2 = b^i b_i.$$

The angular metric tensor defined by $h_{ij} = L\left(\frac{\partial^2 L}{\partial y^i \partial y^j}\right)$ is reducible to

$$h_{ij} = \tau(a_{ij} - Y_i Y_j). \qquad (2.2)$$

Differentiating (2.1) partially with respect to y^k, we have

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{2L}\left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right), \qquad (2.3)$$

Where, $m_i = b_i - \frac{\beta}{\alpha^2} y_i = b_i - \mu Y_i$. From the relation (2.3), we have

$$C^{i}_{jk} = \frac{1}{2L}\left(h^{i}_{j} m_k + h^{i}_{k} m_j + h_{jk} m^i\right), \qquad (2.4)$$

Where, $h^{i}_{j} = g^{ir} h_{rj}, \qquad m^i = g^{ir} m_r$

In Randers space F^n, the Riemannian space $R^n = (M^n, L)$ is called associated Riemannian space with F^n. The Christoffel symbols of R^n are denoted by $\begin{Bmatrix} i \\ jk \end{Bmatrix}$. We assume that ∇_k stands for the covariant differentiation with respect to x^k, relative to associated Riemannian connection, we put

$$b_{jk} = \nabla_k b_j = \partial_k b_j - b_r \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

$$E_{jk} = \frac{b_{jk}+b_{kj}}{2} = b_{(jk)} \tag{2.5}$$

$$F_{jk} = \frac{b_{jk} - b_{kj}}{2} = b_{[jk]}$$

Then a straight forward calculation leads us to

$$\gamma^k_{ij} = g^{kr}\gamma_{irj} = g^{kr}(\partial_j g_{ir} + \partial_i g_{rj-}\partial_r g_{ij})/2,$$

$$= \begin{Bmatrix} k \\ ij \end{Bmatrix} + l^i E_{ij} + l_i F^k_j + l_j F^k_i + \begin{Bmatrix} s \\ 0j \end{Bmatrix} C^k_{is} + \begin{Bmatrix} s \\ 0i \end{Bmatrix} C^k_{js} - \begin{Bmatrix} s \\ 0m \end{Bmatrix} g^{mk} C_{ijs} + b_{0j} N^k_i + b_{0i} N^k_j - b_{0m} g^{mk} N_{ij}, \tag{2.6}$$

Where, we put $l_i = Y_i + b_i$, $\quad F^k_i = g^{kr} F_{ri}$.

For the symmetric tensor $N_{ij} = \frac{h_{ij}}{L}$ and covariant vector l_k, we get $N_{i0} = 0$, $l_0 = L$, here the suffix "0" means the contraction by y^i.

Putting $2G^i = \gamma^i_{00} = \gamma^i_{jk} y^j y^k$, we have from (2.6)

$$2G^i = \begin{Bmatrix} i \\ 00 \end{Bmatrix} + l^i E_{00} + 2LF^i_0 \tag{2.7}$$

The non-linear connection $G^i_j = \partial_j G^i$ is obtained as follows

$$G^i_j = \begin{Bmatrix} i \\ j0 \end{Bmatrix} + l^i E_{j0} + l_j F^i_0 + \left(N^i_j - l^m C^i_{mj}\right) E_{00} + L(F^i_j - 2F^m_0 C^i_{jm}) \tag{2.8}$$

In the Berwald h-connection $G^i_{jk} = \partial_k G^i_j$ of the Randers space, we get

$$G^i_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + l^i E_{jk} + l_k F^i_j + l_j + 2\left(N^i_j E_{k0} + N^i_k E_{j0} + N_{jk} F^i_0\right) + g^{im} N_{mj(k)} E_{00} - 2S_{(jk)}\left\{C^i_{mj} A^m_k\right\} + \lambda^s (2C^i_{jm} C^m_{sk} - C^i_{sj(k)}), \tag{2.9}$$

Where, we put

$$A^m_k = N^m_k E_{00} + l^m E_{k0} + l_k F^m_0 + LF^m_k,$$

$$\lambda^s = l^s E_{00} + 2LF^s_o,$$

$C^i_{sj(k)} = \partial_k C^i_{sj},$

$N_{mj(k)} = \partial_k N_{mj},\ S_{(ij)}\{C^i_{mj}A^m_k\} = C^i_{mj}A^m_k + C^i_{mk}A^m_j.$

Now equation (2.7) can be rewritten as

$$G^i = \begin{Bmatrix} i \\ 00 \end{Bmatrix} + \frac{y^i}{L}E_{00} + 2\alpha a^{ir}F_{r0} - 2\alpha\frac{y^i}{L}b^r F_{r0}. \qquad (2.10)$$

From the above equation, we obtain

$$G^i_j = 2\begin{Bmatrix} i \\ j0 \end{Bmatrix} + \left[2E_{j0} - \frac{E_{00}}{L}\left(\frac{a_{j0}}{\alpha} + b_j\right) - 2\frac{a_{j0}}{\alpha}F_0 + 2\alpha F_j - 2\alpha F_0\left(\frac{a_{j0}}{\alpha} + b_j\right)\right] + \left(\delta^i_j\frac{E_{00}}{L} + 2\frac{a_{j0}}{\alpha}a^{ir}F_{r0} + 2\alpha a^{ir}F_{rj} - \frac{2\alpha}{L}F_0\delta^i_j\right). \qquad (2.11)$$

After contraction (2.11) by the indices i, j and differentiating this equation by $y^k \& y^l$, we get the following

$$G_{kl} = \frac{(n+2)}{L}\left[\frac{E_{00}}{L}\left(\frac{a_{0k}a_{l0}}{\alpha^3} - \frac{2}{L}\left(\left(\frac{a_{0k}}{2} + b_k\right)\left(\frac{a_{0l}}{\alpha} + b_l\right)\right)\right) - \frac{2}{L}\left(E_{0k}\left(\frac{a_{0l}}{\alpha} + b_l\right) + E_{l0}\left(\frac{a_{0k}}{\alpha} + b_k\right)\right)\right]$$

Hence from the structure of the above equation, we have

*Theorem 2.1.*in a n-dimensional Randers space if $\mathrm{E_{kl}} = 0$then $\mathrm{F^n}$is weakly Berwald space.

The Cartan h-connection Γ^{*i}_{jk} of the Randers space is well known [12] as follows:

$$\Gamma^{*i}_{jk} = \gamma^i_{jk} + g^{im}C_{jkr}G^r_m - C^i_{kr}G^j_r - C^i_{jr}G^r_k,$$

$$= \begin{Bmatrix} i \\ jk \end{Bmatrix} + l^iE_{jk} + l_jF^i_k + +l_kF^i_j + b_{0k}N^i_j + b_{0j}N^i_k - \left(C^i_{kr}A^r_j + C^i_{jr}A^r_k - g^{is}C_{jkr}A^m_s\right) - b_{0m}g^{mi}N_{jk} + \lambda^s\left(C^i_{km}C^m_{sj} + C^i_{km}C^m_{sj} - C^m_{kj}C^i_{ms}\right) \qquad (2.12)$$

Form (2.12), the difference tensor of the Cartan connection $C\Gamma$ is given [10] as follows:

$$D^i_{jk} = \Gamma^{*i}_{jk} - \begin{Bmatrix} i \\ jk \end{Bmatrix}$$

$$= l^iE_{jk} + l_jF^i_k + l_kF^i_j + b_{0k}N^i_j + b_{0j}N^i_k - \left(C^i_{kr}A^r_j + C^i_{jr}A^r_k - g^{is}C_{jkr}A^m_s\right) - b_{0m}g^{mi}N_{jk} + \lambda^s\left(C^i_{km}C^m_{sj} + C^i_{km}C^m_{sj} - C^m_{kj}C^i_{ms}\right),$$

Next from (2.9) the difference tensor of Berwald Connection $\beta\Gamma$ is given by

$${}'D^i_{jk} = G^i_{jk} - \left\{ {i \atop jk} \right\},$$

$$= l^i E_{jk} + l_k F^i_j + l_j F^i_k + g^{im} N_{mj(k)} E_{00} + 2\left(N^i_j E_{k0} + N^i_k E_{j0} + N_{jk} F^i_0\right) - 2S_{(jk)}\left\{C^i_{mj} A^m_k\right\} + \lambda^s\left(2C^i_{jm} C^m_{sk} - C^i_{sj(k)}\right).$$

$$= l^i E_{jk} + l_k F^i_j + l_j F^i_k + g^{im} N_{mj(k)} E_{00} + 2\left(N^i_j E_{k0} + N^i_k E_{j0} + N_{jk} F^i_0\right)$$

$$-2S_{(jk)}\left\{C^i_{mj} A^m_k\right\} + \lambda^s\left(2C^i_{jm} C^m_{sk} - C^i_{sj(k)}\right).$$

On C-conformal change of Rander space

Consider two Rander spaces F^n and $\bar{F}^n$ represented by the same co-ordinate system. Let R^n and $\bar{R}^n$ be the associated Riemannian spaces with the metric tenso $a_{ij}(x)$ and $\bar{a}_{ij}(x)$ respectively.

Now we have the relations,

$$\bar{a}_{ij} = e^{2\sigma} a_{ij}, \qquad \bar{b}_i = e^{\sigma} b_i, \tag{3.1}$$

Where $\sigma = \sigma(x)$ is a scalar function. Under C-Conformal transformation of the Rander space $F^n = \left(M^n, L(\alpha, \beta)\right)$, where $L = \alpha + \beta$ we have the following relations

$$\bar{L} = e^{\sigma} L, \qquad \bar{\alpha} = e^{\sigma}\alpha, \qquad \bar{\beta} = e^{\sigma}\beta,$$

$$\bar{l}_i = e^{\sigma} l_i,\ \bar{l}^i = e^{-\sigma} l^i, \bar{h}_{ij} = e^{2\sigma} h_{ij},\ \bar{h}^i_j = h^i_j, \qquad \bar{g}_{ij} = e^{2\sigma} g_{ij},\ \bar{g}^{ij} = e^{-2\sigma} g_{ij}, \tag{3.2}$$

$$\bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \qquad \bar{C}^i_{jk} = C^i_{jk},$$

$$\bar{\mu} = \mu, \qquad \bar{m}_i = e^{\sigma} m_i,$$

Now taking covariant derivative of $\bar{b}_i$ with respect to x^j in $\bar{R}^n$, we have

$$\nabla_J \bar{b}_i = \bar{b}_{ij} = \frac{\partial \bar{b}_i}{\partial x^j} - \bar{b}_k \overline{\left\{ {\iota \atop Jk} \right\}}. \tag{3.3}$$

Where, we used the relation

$$\overline{\left\{ {\iota \atop Jk} \right\}} = \left\{ {i \atop jk} \right\} + \delta^k_i \sigma_j + \delta^k_j \sigma_i - a_{ij} a^{km} \sigma_m, \tag{3.4}$$

Where, $\sigma_i = \frac{\partial \sigma}{\partial x^i}$ with the help of relations (3.1) and (3.2), equation (2.5) reduced to

$$\bar{b}_{ij} = e^{\sigma}\left(b_{ij} - \sigma_i b_j + a_{ij} b^m \sigma_m\right), \tag{3.5}$$

$$\bar{E}_{ij} = e^{\sigma}\left(E_{ij} - \sigma_i b_j + a_{ij} \sigma_m b_m\right),$$

$\bar{E}_{00} = e^{\sigma}(E_{00} - \sigma_0 b_0 + \alpha^2 \sigma_m b_m)$,

$\bar{F}_{ij} = e^{\sigma}\left(F_{ij} - \sigma_i b_j\right)$,

$\bar{F}_i^k = e^{-\sigma}\left(F_i^k - g^{kr}\sigma_r b_i\right)$.

If we assume that $\bar{b}_{ij} = b_{ij} = 0$, then the relation (3.5) after simplification, we get $b^m \sigma_m = 0$ for $n > 1$. Since $\sigma \neq 0$, then we have

Theorem 3.1. If $\mathrm{n} > 1$ the vecor field $\mathrm{b_i}$ is parallel with respect to the associated Riemannian connection and the vector field $\bar{\mathrm{b}}_\mathrm{i}$ is parallel with respect to the C-Confomally transformed associated Riemannian connection then the vector $\mathrm{b^i}$ is orthogonal to σ_i.

Now we shall find the connection coefficients of the Cartan and Berwald connections of a C-Conformal transformed Randers space. The C-Conformal transformation of (2.7) is reduces to the following forms

$$\bar{G}^i = G^i - B^{ir}\sigma_r, \tag{3.6}$$

where

$$B^{ir} = \left\{\alpha^2 a^{ir} - 2y^i y^r + l^i(y^r b_0 - \alpha^2 b^r) + Lg^{il}(\delta_l^r b_0 - y^r b_l)\right\}/2.$$

Next differentiating (3.6) with respect to y^i, we have

$$\bar{G}_j^i = \partial_j G^i - \partial_j\left(B^{ir}\sigma_r\right),$$

$$= G_j^i - B_j^{ir}\sigma_r, \tag{3.7}$$

Where,

$$B_j^{ir} = a^{ir}Y_j - \left(\delta_j^i y^r + \delta_j^r y^i\right)l^i \left\{\frac{\left(\delta_j^r b_0 + y^r b_j\right)}{2} - Y_j b^r\right\} + N_j^i(y^r b_0 - \alpha^2 b^r) + l_j g^{il}\left(\frac{\delta_l^r b_0 - y^r b_l}{2}\right) + Lg^{il}\delta_l b_j.$$

Furthermore differentiating (3.7) with respect to y^k, we obtain

$$\bar{G}_{jk}^i = G_{jk}^i + \sigma_r\left(B_{jk}^{ir}\right), \tag{3.8}$$

where

$$B_{jk}^{ir} = Q_{jk}^{ir} + l^i\left\{\delta_j^r b_k - a_{jk}b^r\right\} + S_{(jk)}\left\{\delta_j^r b_0 + y^r b_j - 2Y_j b^r + l_j g^{il}\delta_l^r b_k\right\} + N_{jk}g^{il}\left(\delta_l^r b_0 - y^r b_j\right) + \frac{l_k}{2}\left(a_{lj} - l_l l_j\right) - \frac{L}{\alpha^2}(a_{lk} - l_l l_k)l_j + \begin{Bmatrix} i \\ jk \end{Bmatrix}(y^r b_0 - \alpha^2 b^r), \tag{3.9}$$

$$Q_{jk}^{ir} = a_{jk}a^{ir} - \delta_k^i \delta_j^r - \delta_j^i \delta_k^r. \tag{3.10}$$

Therefore, we have the following

Theorem 3.2. Under C-conformal transformation of the Randers space, the connection co-efficient of G^i_j, G^i_{jk} of a Berwald connection βΓ are transformed as (3.7) and (3.8) respectively.

Next, we shall calculate the C-Conformal transformed Quantity $\bar{\Gamma}^{*i}_{jk}$ of the Cartan connection co-efficients Γ^{*i}_{jk}. Using (3.2) and (3.7), we can see that (2.12) is transformed to the following form

$$\bar{\Gamma}^{*i}_{jk} = \overline{\left\{ {i \atop jk} \right\}} + \bar{l}^i \bar{E}_{jk} + \bar{l}_j \bar{F}^i_k + \bar{l}_k \bar{F}^i_j + \bar{b}_{0k} \bar{N}^i_j + \bar{b}_{0j} \bar{N}^i_k - \bar{b}_{0m} \bar{g}^{mi} \bar{N}_{jk} - \left(\bar{C}^i_{kr} \bar{A}^r_j + \bar{C}^i_{jr} \bar{A}^r_k - \bar{g}^{is} \bar{C}_{jkr} \bar{A}^m_s\right) + \bar{\lambda}^s \left(\bar{C}^i_{km} \bar{C}^m_{sj} + \bar{C}^i_{km} \bar{C}^m_{sj} - \bar{C}^m_{kj} \bar{C}^i_{ms}\right), \quad (3.11)$$

Where,

$$U^{ir}_{jk} = Q^{ir}_{jk} + \left(a_{jk} b^r - \frac{\left(\delta^r_j b_k + \delta^r_k b_j\right)}{2} \right) l^i + l_j g^{ir} \left(\frac{\delta^r_k b_r - b_k}{2} \right) + l_k \left(\frac{\delta^r_j b_r - b_j}{2} \right) + S_{(jk)} \left(N^i_j (Y_k b^r - y^r b_k) \right) + g^{im} N_{jk} (y^r b_m - Y_m b^r), \quad (3.12)$$

$$\bar{A}^m_k = A^m_k + \sigma_r \left[(\alpha^2 b^r - y^r b_0) N^m_k + \left(a_{k0} b^r - (\delta^r_k b_0 + y^r b_0) \right) l^m + g^{mr} \left(\frac{y^r b_r - b_0}{2} \right) l_k + \left(\frac{\delta^r_k b_r - b_k}{2} \right) L \right],$$

Form the difference of (3.8) and (3.4) we have

$${}'\bar{D}^i_{jk} = \bar{G}^i_{jk} - \left\{ {i \atop jk} \right\},$$

$$= D^i_{jk} - {}'T^{ir}_{jk} \sigma_r,$$

where ${}'T^{ir}_{jk} = B^{ir}_{jk} - Q^{ir}_{jk}$,

From (3.9) we have

$${}'T^{ir}_{jk} = l^i \{ \delta^r_j b_k - a_{jk} b^r \} + S_{(jk)} \{ \delta^r_j b_0 + y^r b_j - 2Y_j b^r + l_j g^{il} \delta^r_l b_k \} + N_{jk} g^{il} \left(\delta^r_l b_0 - y^r b_j \right) + \frac{l_k}{2} \left(a_{lj} - l_l l_j \right) - \frac{L}{\alpha^2} (a_{lk} - l_l l_k) l_j + \left\{ {i \atop jk} \right\} (y^r b_0 - \alpha^2 b^r)$$

Thus we have

Theorem 3.3. A difference tensor ${}'\bar{D}^i_{jk}$ of the Berwald connection of the Randers space is invariant under C-Conformal transformation if and only if ${}'T^{ir}_{jk} \sigma_r = 0$.

Now we shall calculate the C-Conformal transformation of difference tensor D^i_{jk} of $C\Gamma$.

From (3.11) and (3.5) we have

$$\bar{D}^i_{jk} = \bar{\Gamma}^{*i}_{jk} - \overline{\left\{ {i \atop jk} \right\}},$$

$$= D^i_{jk} - T^{ir}_{jk}\sigma_r. \qquad (3.13)$$

where $T^{ir}_{jk} = U^{ir}_{jk} - Q^{ir}_{jk}$.

From (3.13) we get

$$T^{ir}_{jk} = \left(a_{jk}b^r - \frac{\left(\delta^r_j b_k + \delta^r_k b_j\right)}{2}\right) l^i + l_j g^{ir}\left(\frac{\delta^r_k b_r - b_k}{2}\right) + l_k\left(\frac{\delta^r_j b_r - b_j}{2}\right) + S_{(jk)}\left(N^i_j(Y_k b^r - y^r b_k)\right) + g^{im}N_{jk}(y^r b_m - Y_m b^r)$$

Theorem 3.4. The difference tensor $\mathrm{D}^{\mathrm{i}}_{\mathrm{jk}}$ of the Cartan connection is invariant under C-Confomal transformation of the Rander space if and only if $\mathrm{T}^{\mathrm{ir}}_{\mathrm{jk}} = 0$.

The following lemma has been proved by C.Shibata, H.Shimada, M.Asuma and H.Yasuda [12]

Lemma 3.1. The difference tensor D^i_{jk} vanishes if and only if the covariant vector field b_i is parallel with respect to the associate Riemannian connection.

In vive of the above lemma and the relation (3.13), we have the following

Theorem 3.5. If the vector field b_i is parallel in the associated Riemannian space R^n then the vector field b_i is parallel in the C-conformal transformed associated Riemannian space $\bar{R}^n$ if and only if $T^{ir}_{jk}\sigma_r$ vanishes identically..

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