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FIXED POINT THEOREM AND SUB-COMPATIBILITY IN FUZZY METRIC SPACE

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Abstract

In this paper, the concept of sub-compatibility in fuzzy metric space has been applied to prove a common fixed point theorem for six self maps using implicit relation. Our result generalizes and extends the result of Ranadive and Chouhan [13].

Keywords: Fuzzy metric space, common fixed point, absorbing maps, sub-compatibility, and sub-sequential continuity.

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1. Introduction.

The theory of fuzzy sets was introduced by Zadeh [18] in 1965. Zadeh [19] estimated that medical diagnosis would be the most liable application domain of Fuzzy set theory. George and Veeramani [4] and Kramosil and Michalek [7] have introduced the concept of fuzzy metric spaces which can be regarded as a simplification of the statistical (probabilistic) metric space. Afterwards, Grabiec [5] defined the completeness of the fuzzy metric space. Following Grabiec's work, Fang [3] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra et. al. [8] also obtained numerous common fixed point theorems for asymptotically commuting maps in the same space, which generalize a number of fixed point theorems in metric, Menger, fuzzy and uniform spaces.

The concepts of semi-compatibility and weak-compatibility in fuzzy metric space were given by Singh and Jain [15] which was simplification of commuting and compatible maps. Popa [10, 11] introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. Singh and Jain [16] further extended the result of Popa [10-11] in fuzzy metric spaces. Using the concept of R-weak commutative mappings, Vasuki [17] proved the fixed point theorems for fuzzy metric space. In 2009, using the concept of sub-compatible maps, Bouhadjera et. al. [2] proved common fixed point theorems. In 2010 and 2011, Singh et. al. [14, 16] proved fixed point theorems in fuzzy metric space using the concept of semi-compatibility, weak compatibility and compatibility of type (β) respectively. Ranadive et.al. [13] introduced the concept of absorbing mapping in fuzzy metric space and proved the common fixed point theorem in this space. Moreover, Ranadive et.al. [13] observed that the new notion of absorbing map is neither a sub class of compatible maps nor a subclass of non compatible maps. Afterwards, Mishra et. al. [9] proved fixed point theorems using absorbing mappings in fuzzy metric space.

2. Preliminaries.

Definition 2.1. [7] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min (a, b)$

Definition 2.2. [7] The three tuple (X, M,*) is called a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions:

for all x, y, z $\in$ X and s,t > 0,

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) =1, for all t > 0 if and only if x = y

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) M(x, y, t)* M(y, z, s) $\geq$ M(x, z, t+s)

(FM-5) M(x, y, .) : $[0, \infty) \rightarrow [0, 1]$ is left continuous.

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$.

Example 2.1.[7] Let (X,d) be a metric space. Define a*b = min{a,b} and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all x, y $\in$ X and all t > 0. Then (X, M, *) is a fuzzy metric space. It is called the fuzzy metric space induced by d.

Definition 2.3. [7] A sequence $\{x_n\}$ in a Fuzzy metric space (X,M,*) is said to be a Cauchy sequence if and only if for each $\varepsilon > 0$, t > 0 there exists $n_0 \in N$ such that $M(x_n,x_m,t) > 1 - \varepsilon$ for all n, m $\geq n_0$.

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each $\varepsilon > 0$, t > 0 there exists $n_0 \in$ N such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A fuzzy metric space (X,M,*) is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [1] A pair (A, B) of self maps of a fuzzy metric space (X, M, *) is said to be reciprocal continuous if $\lim_{n\to\infty} ABx_n = Ax$ and $\lim_{n\to\infty} BAx_n = Bx$ whenever there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$. If A and B are both continuous then they are obviously reciprocally continuous but the converse need not be true.

Definition 2.5. [15] Let A and B be mappings from fuzzy metric space (X, M, *) into itself. The mappings A and B are said to be compatible if and only if $M(ASx_n, SAx_n, t) \rightarrow 1$, for all t > 0 whenever $\{x_n\}$ is a sequence in X such that
$Sx_n, Ax_n \rightarrow p$ for some p in X as $n \rightarrow \infty$.

Definition 2.6. [15] Let A and S be mappings from fuzzy metric space (X,M,*) in to itself. Then the mappings A and S are said to be semi-compatible if

$$\lim_{n\to\infty} ASx_n = Sx,$$

whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

It follows that if (A,S) is semi compatible and Ay = Sy, then ASy = SAy by taking
$\{x_n\} = y$ and x = Ay = Sy.

Definition 2.7. [9]. A pair of maps A and B is called weakly compatible pair if they commute at their coincidence points i.e. Ax = Bx if and only if ABx = BAx.

Definition 2.8. [13]. Let A and B be two self maps on a fuzzy metric space (X, M, *) then A is called B-absorbing if there exists a positive integer R > 0 such that M(Bx, BAx, t) ≥ M(Bx, Ax, t/R) for all x ∈ X.

Similarly B is called A-absorbing if there exists a positive integer R > 0 such that M(Ax, ABx, t) ≥ M(Ax, Bx, t/R) for all x ∈ X.

Preposition 2.1. In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Preposition 2.2. [9] If (A,S) is a semi compatible pair of self maps of a fuzzy metric space (X, M, *) and S is continuous, then (A,S) is compatible.

Lemma 2.1. [8] Let (X, M, *) be a fuzzy metric space. Then for all x, y ∈ X, M(x, y, .) is a non-decreasing function.

Lemma 2.2. [8] Let (X, M, *) be a fuzzy metric space. If there exists k ∈ (0, 1) such that for all x, y ∈ X, M(x, y, kt) ≥ M(x, y, t) for all t > 0, then x = y.

Lemma 2.3. [8] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k ∈ (0, 1) such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$, for all t > 0 and n ∈ N. Then $\{x_n\}$ is a Cauchy sequence in X.

Preposition 2.3. [6] Let A and B be mappings from a fuzzy metric space (X, M, *) into itself. Assume that (A, B) is reciprocal continuous then (A, B) is semi-compatible if and only if (A, B) is compatible.

Definition 2.9. [6] Self mappings A and S of a fuzzy metric space (X, M, *) are said to be sub-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, z □ X and satisfy $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$.

Clearly, semi-compatible maps are sub-compatible maps but converse is not true.

Example 2.2. Let X = $[0,\infty)$ with usual metric d and define $M(x,y,t) = \dfrac{t}{t + d(x,y)}$ for all x, y□ X , t > 0 define the self maps A, S as

$$Ax = \begin{cases} 2 + x, & 0 \le x \le 2 \\ 3x - 1, & 2 < x < \infty \end{cases} \quad \text{and} \quad Sx = \begin{cases} 2 - x, & 0 \le x \le 2 \\ 3x - 2, & 2 < x < \infty \end{cases}.$$

Define a sequence $\{x_n\} = \dfrac{2}{n}$ in X. Then

$$Ax_n = 2 + \frac{2}{n} \quad \text{and} \quad Sx_n = 2 - \frac{1}{n}.$$

Also, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \lim_{n\to\infty} M(4, 4, t) = 1$.

Now, $\lim_{n\to\infty} Ax_n = 2$ and $\lim_{n\to\infty} Sx_n = 2$

This implies $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$. But $\lim_{n\to\infty} ASx_n \neq Sx$.

Thus, A and S are sub-compatible but not semi-compatible.

Definition 2.10. [13] A class of implicit relation

Let □ be the set of all real continuous functions F : $(R^+)^5$ □ R non-decreasing in first argument satisfying the following conditions :

(i) For u, v □ 0, F(u, v, v, u, 1) □ 0 implies that u □ v.

(ii) F(u, 1, 1, u, 1) □ 0 or F(u, 1, u, 1, u) □ 0, or F(u, u, 1, 1, u) □ 0 implies that u□□□1.

Example 2.4. Define $F(t_1, t_2, t_3, t_4, t_5) = 16t_1 - 12t_2 - 8t_3 + 4t_4 + t_5 - 1$. Then F□□□□.

(i) F(u, v, v, u, 1) = 20(u - v) □ 0 □ u □□ v.

(ii) F(u, 1, 1, u, 1) = 20(u - 1) □ 0 □ u □ 1 or

F(u, 1, u, 1, u) = 9(u - 1) □□ 0 □ u □ 1

or F(u, u, 1, 1, u) = 5(u - 1) □ 0 □ u □ 1.

3. Main Result

Theorem 3.1. Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, *) with t-norm defined by a * b = min{a, b}, satisfying :

(3.1) P(X) □ ST(X), Q(X) □ AB(X);

(3.2) Q is ST-absorbing;

(3.3) for some F□□□ there exists q □□(0,1) such that for all x, y □□X and $t > 0$

$$F\{M(Px, Qy, qt), M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, qt),$$

$$M(Px, STy, t)\} \geq 0.$$

(3.4) AB = BA, ST = TS, PB = BP, QT = TQ.

If the pair of maps (P, AB) is reciprocal continuous and sub-compatible then P, Q, S, T, A and B have a unique common fixed point in X.

Proof. Let x_0 □X be any arbitrary point. From (3.1), there exist x_1, x_2 □ X such that

$$Px_0 = STx_1 \quad \text{and} \quad Qx_1 = ABx_2 .$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and

$Qx_{2n-1} = ABx_{2n} = y_{2n}$ for $n = 1, 2, 3, \ldots$.

Step 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.3), we get

$F\{M (Px_{2n}, Qx_{2n+1}, qt), M (ABx_{2n}, STx_{2n+1}, t), M (Px_{2n}, ABx_{2n}, t),$

$M(Qx_{2n+1}, STx_{2n+1}, qt), M(Px_{2n}, STx_{2n+1}, t)\} \geq 0,$

i.e., $F\{M(y_{2n+1}, y_{2n+2}, qt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, qt),$

$M(y_{2n+1}, y_{2n+1}, t)\} \geq 0.$

Using lemmas 2.1 and 2.2, we have

$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t).$

Again substituting $x = x_{2n+2}$ and $y = x_{2n+3}$ in (3.3), we get

$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t).$

Hence by lemma 2.3, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, therefore,
$\{y_n\} \to z$ in X and also its subsequences converges to the same point i.e. z X,

i.e. $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$ (1)

$\{Px_{2n}\} \to z$ $\{ABx_{2n}\} \to z$ (2)

Step 2. (P, AB) is sub-compatible and reciprocally continuous then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = z, \; z \in X \quad \text{and satisfy}$$

$$\lim_{n\to\infty} M(P(AB)x_n, (AB)Px_n, t) = M(Pz, ABz, t) = 1.$$

Therefore , $Pz = ABz.$ (3)

Step 3. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (3.3), we have

$F\{M(PPx_{2n}, Qx_{2n+1}, qt), M(ABPx_{2n}, STx_{2n+1}, t), M(PPx_{2n}, ABx_{2n}, t),$

$M(Qx_{2n+1}, STx_{2n+1}, qt), M(PPx_{2n}, STx_{2n+1}, t)\} \geq 0$

Taking $n\to\infty$ and using (1), (2), (3), we get

$F\{M(Pz, z, qt), M(Pz, z, t), M(Pz, Pz, t), M(z, z, qt), M(Pz, z, t)\} \geq 0$

$F\{M(Pz, z, qt), M(Pz, z, t)\} \geq 0$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$

Therefore by using lemma 2.2, we have

$z = Pz = ABz$

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (3.3), we get,

$$F\{M(PBz, Qx_{2n+1}, qt), M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t), M(Qx_{2n+1}, STx_{2n+1}, qt), M(PBz, STx_{2n+1}, t)\} \geq 0$$

As $BP = PB$, $AB = BA$, so we have

$P(Bz) = B(Pz) = Bz$ and $(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.$

Taking $n \to \infty$ and using (1), we get

$$F\{M(Bz, z, qt), M(Bz, z, t), M(Bz, Bz, t), M(z, z, qt), M(Bz, z, t)\} \geq 0$$

$$F\{M(Bz, z, qt), M(Bz, z, t)\} \geq 0$$

i.e., $M(Bz, z, qt) \geq M(Bz, z, t)$.

Therefore by using lemma 2.2, we have

$Bz = z$ and also we have $ABz = Z$

This implies $Az = z$

Therefore $Az = Bz = Pz = z$. (4)

Step 5. As P(X) □□ST(X), there exist u □ X such that

$z = Pz = STu$. (5)

Putting $x = x_{2n}$ and $y = u$ in condition (3.3), we get

$$F\{M(Px_{2n}, Qu, qt), M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t), M(Qu, STu, qt), M(Px_{2n}, STu, t)\} \geq 0.$$

Letting $n \to \infty$ and using (2) and (5), we get

$$F\{M(z, Qu, qt), M(z, z, t), M(z, Pz, t), M(Qu, z, qt), M(z, z, t)\} \geq 0$$

As F is non-decreasing in the first argument, we have

$F\{M(z, Qu, qt), 1, 1, M(Qu, z, qt), 1\} \geq 0$

i.e., $M(z, Qu, qt) \geq 1$.

Therefore, $z = Qu = STu$.

Since Q is ST absorbing, we have

$M(STu, STQu, t) \geq M(STu, Qu, t/R) \geq 1$

i.e., $STu = STQu$ which implies $z = STz$.

Putting $x = z$ and $y = z$ in (3.3) , we get

$F\{M(Pz, Qz, qt), M(ABz, STz, t), M(Pz, ABz, t), M(Qz, STz, qt), M(Pz, STz, t)\} \geq 0$

or, $F\{M(z, Qz, qt), M(z, z, t), M(z, z, t), M(Qz, z, qt), M(z, z, t)\} \geq 0$.

As F is non-decreasing in the first argument, we have

$F\{M(z, Qz, qt), 1, 1, M(Qz, z, qt), 1\} \geq 0$,

i.e., $M(z, Qz, qt) \geq 1$.

Therefore, $z = Qz$

Hence, $z = Qz = STz$.

Step 6. Putting $x = x_{2n}$ and $y = Tz$ in condition (3.3), we get

$$F\{M(Px_{2n}, QTz, qt), M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, qt), M(Px_{2n}, STTz, t)\} \geq 0$$

As $QT = TQ$ and $ST = TS$, we have

$QTz = TQz = Tz$ and $ST(Tz) = T(STz) = TQz = Tz$.

Letting $n \to \infty$ and using (2) we get

$F\{M(z, Tz, qt), M(z, Tz, t), M(z, z, t), M(Tz, Tz, qt), M(z, Tz, t)\} \geq 0$

$F\{M(z, Tz, qt), M(z, Tz, t)\} \geq 0$

i.e., $M(z, Tz, qt) \geq M(z, Tz, t)$.

Therefore, by lemma 2.2, we get

Tz = z

Now, STz = Tz = z implies Sz = z.

Hence, Sz = Tz = Qz = z. (7)

Hence, z is the common fixed point of A, B, S, T, P and Q.

Uniqueness: Let w be another fixed point of A, B, P, Q, S and T. Then putting x = z and y = u in (3.3), we get

F{M (Pz, Qu, qt), M (ABz, STu, t), M (Pz, ABz, t),

M (Qu, STu, qt), M (Pz, STu, t)} $\geq$ 0

As F is non-decreasing in the first argument, we have

F{M(z, u, qt), M(z, u, t), M(z, z, t), M(u, u, qt), M(z, u, t)} $\geq$ 0

or, F{M(z, u, qt), M(z, u, t), 1, 1, M(z, u, t)} $\geq$ 0

i.e. z = u.

Hence z is unique fixed point in X.

Remark 3.1. If we take B = T = I (the identity map) in theorem 3.1, we get the following corollary.

Corollary 3.1. Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, *) with t-norm defined by a * b = min{a, b}, satisfying :

(3.1) P(X) □ S(X), Q(X) □ A(X);

(3.2) Q is S-absorbing;

(3.3) for some F□□□ there exists k □□(0,1) such that for all x, y □□X and t > 0

F{M(Px, Qy, kt), M(Ax, Sy, t), M (Px, Ax, t), M(Qy, Sy, kt), M(Px, Sy, t)} $\geq$ 0.

If the pair of maps (P, A) is reciprocal continuous and sub-compatible then P, Q, S and A have a unique common fixed point in X.

Remark 3.2. In view of Remark 3.1, Corollary 3.1 is a generalization of the result of Ranadive and Chouhan [13] in the sense that condition of semi-compatible maps has been replaced by sub-compatible maps.

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