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Explicit Iterative Algorithms to Solve Generalized System of Nonlinear Mixed Variational Inequalities

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Abstract.- The objective of this paper is to consider new generalized systems of nonlinear mixed variational inequalities including $3k$-distinct nonlinear relaxed cocoercive operators. We proposed k-steps explicit iterative methods including resolvent operators for this considered system and present the equivalent fixed point problem of this new generalized systems of variational inequalities. This equivalent fixed point problem suggest us k-steps explicit iterative algorithms to obtain approximate solution of considered system. Convergence results of k-step explicit iterative algorithms are obtained.

1. Introduction.

In this paper, we consider and study the generalized system of nonlinear mixed variational inequalities. There are several techniques to solved variation inequality problems. The resolvent operator technique is one of the iterative methods to solve variational inequalities. The resolvent operator technique is the generalized form of projection methods. Many heuristics widely used projection techniques to solve variational inequalities and systems of variational inequalities. Recently Noor et. al [6, 9], Verma [11, 12], Hao et. al [2], Kim [4] and Zhang [13] obtained the approximate solution of system of nonlinear variational inequalities by using two or three steps iterative methods involving projection operator. For further details, please see [1, 2, 4, 7, 8, 9, 11, 12, 13]. Very recently, Noor and Noor [6], Husain and Gupta [3] and Kim and Kim [5], put forward two steps iterative methods linked with resolvent operator to establish the convergence result.

This present work is impelled by the research going on this field. The aim is to perusal the new generalized system of nonlinear variational inequalities connecting with $3k$-distinct nonlinear relaxed (r,s)-cocoercive operators. First, we give the fixed point problem equivalent to considered system. By this equivalent formulations, we proposed k-steps explicit algorithms with resolvent operator. Utilization of resolvent operator approach, we make an attempt to obtain an approximate solution of the generalized system of nonlinear variational inequalities. The consider conditions guaranteed the convergence of iterative sequences obtained by the k-steps explicit algorithms. This work extend and improve the well-known results in the literature [2, 3, 4, 5, 6, 7, 11, 12, 13].

Throughout the manuscript, H is a real Hilbert space endowed with a norm $\|.\|$ and an inner product $\langle .,. \rangle$. Let $D \subset H$ be a closed and convex set in H.

Let us given that nonlinear operators $A, f, g: H \rightarrow H$ and continuous function $\varphi: H \rightarrow \mathbb{R} \cup \{+\infty\}$, then generalized nonlinear mixed variational inequality problem (GNMVIP) is to find $p^* \in H$ such that

$$\langle \lambda A(p^*) + g(p^*) - f(p^*), p - g(p^*) \rangle \geq \varphi(g(p^*)) - \varphi(p), \forall\, p \in H, \lambda > 0. \qquad (1.1)$$

It observe that p^* is a solution of GNMVIP (1.1) if and only if p^* is a fixed point of $I - g - J_\varphi[f - \lambda A]$.

If $f = g = I$, then the GNMVIP (1.1) is equivalent to find $p^* \in H$ such that

$$\langle A(p^*), p - p^* \rangle \geq \varphi(p^*) - \varphi(p), \forall\, p \in H. \qquad (1.2)$$

The resolvent operator connected with maximal monotone operator A is given as

$$J_A(p^*) = (I + \lambda A)^{-1}(p^*), \quad \forall\, p^* \in H, \lambda > 0.$$

If A is maximal monotone if and only if its resolvent operator J_A connected with A is single-valued as well as non-expansive. If $\varphi(.)$ is a proper, convex and lower-semicontinuous function, then its subdifferential $\partial\varphi(.)$ is a maximal monotone operator. Then, resolvent operator J_φ connected with $\partial\varphi$ is given as

$$J_\varphi(p^*) = (I + \lambda \partial\varphi)^{-1}(u), \forall\, p^* \in H.$$

Lemma 1.1 *For given $p^*, q^* \in H$ satisfies the inequality*

$$\langle p^* - q^*, p - p^* \rangle \;\geq\; \lambda\varphi(p^*) - \lambda\varphi(p), \;\; p \in H$$

if and only if $p^* = J_\varphi(q^*)$ where $J_\varphi = (I + \lambda\partial\varphi)^{-1}$ is the resolvent operator.

One can easily prove that J_φ is nonexpansive, that is, $\| J_\varphi p^* - J_\varphi p \| \leq \| p^* - p \|, \forall p, p^* \in H$.

If φ is an indicator function of a closed convex set $D \subset H$, then $J_\varphi = P_D$ i.e.

$$\varphi(p^*) = \begin{cases} 0, p^* \in D \\ +\infty, otherwise, \end{cases}$$

then Problem (1.2) reduces to classical variational inequality (1.3) proposed by Stampacchia [10], given as

$$\langle A(p^*), p - p^* \rangle \geq 0, p \in D. \tag{1.3}$$

Definition 1.2 A mapping $A: H \to H$

(i) is μ-strongly monotone if $\exists$ a constant $\mu > 0$ such that

$$\langle Ap^1 - Ap^{1*}, p^1 - p^{1*} \rangle \geq \mu \| p^1 - p^{1*} \|^2, \forall p^1, p^{1*} \in H;$$

(ii) is relaxed $(\bar{r}, \bar{s})$-cocoercive if $\exists$ constants $\bar{r}, \bar{s} > 0$ such that

$$\langle Ap^1 - Ap^{1*}, p^1 - p^{1*} \rangle \geq -\bar{r} \| Ap^1 - Ap^{1*} \|^2 + \bar{s} \| p^1 - p^{1*} \|^2, \forall p^1, p^{1*} \in H;$$

(iii) is $\bar{t}$-Lipschitz continuous if $\exists$ a constant $\bar{t} > 0$ such that

$$\| Ap^1 - Ap^{1*} \| \leq \bar{t} \| p^1 - p^{1*} \|, \forall p^1, p^{1*} \in H.$$

Let $A_i: \underbrace{H \times H \times \ldots \times H}_{k\ times} \to H$ and $f_i, g_i: H \to H$ be $3k$-distinct nonlinear operators for each $i \in \{1,2,\ldots,k\}$. Then generalized system of nonlinear mixed variational inequalities problem (GSNMVIP) is to find $(p^1, p^2, \ldots p^k) \in \underbrace{H \times H \times \ldots \times H}_{(k\ times)}$ such that

$$\begin{cases} \langle \lambda_1 A_1(p^2, p^3, \ldots p^k, p^1) + g_1(p^1) - f_1(p^2), p - g_1(p^1) \rangle \geq \lambda_1 \varphi\big(g_1(p^1)\big) - \lambda_1 \varphi(p), \forall p \in H, \\ \langle \lambda_2 A_2(p^3, p^4, \ldots p^1, p^2) + g_2(p^2) - f_2(p^3), p - g_2(p^2) \rangle \geq \lambda_2 \varphi\big(g_2(p^2)\big) - \lambda_2 \varphi(p), \forall p \in H, \\ \vdots \\ \langle \lambda_{k-1} A_{k-1}(p^K, p^1, \ldots p^{k-2}, p^{k-1}) + g_{k-1}(p^{k-1}) - f_{k-1}(p^k), p - g_{k-1}(p^{k-1}) \rangle \\ \qquad \geq \lambda_{k-1} \varphi\Big(g_{k-1}(p^{k-1})\Big) - \lambda_{k-1} \varphi(p), \forall p \in H, \\ \langle \lambda_k A_k(p^1, p^2, \ldots p^{k-1}, p^k) + g_k(p^k) - f_k(p^1), p - g_k(p^k) \rangle \geq \lambda_k \varphi\Big(g_k(p^k)\Big) - \lambda_k \varphi(p), \forall p \in H, \\ for\ each\ \lambda_i > 0, i \in \{1,2,\ldots,k\}. \end{cases} \tag{1.4}$$

Here, we are given some special cases of GSNMVIP (1.4).

(I) If A_i, for each $i \in \{1,2,\ldots,k\}$, is univariate mapping, then GSNMVIP (1.4) reduces to find $(p^1, p^2, \ldots p^k) \in \underbrace{H \times H \times \ldots \times H}_{(k\ times)}$ such that

$$\begin{cases} \langle \lambda_1 A_1(p^2) + g_1(p^1) - f_1(p^2), p - g_1(p^1)\rangle \geq \lambda_1\varphi(g_1(p^1)) - \lambda_1\varphi(p), \forall p \in H, \\ \langle \lambda_2 A_2(p^3) + g_2(p^2) - f_2(p^3), p - g_((p^2)\rangle \geq \lambda_2\varphi(g_2(p^2)) - \lambda_2\varphi(p),\ \forall p \in H, \\ \vdots \\ \langle \lambda_{k-1} A_{k-1}(p^K) + g_{k-1}(p^{k-1}) - f_{k-1}(p^k), p - g_{k-1}(p^{k-1})\rangle \geq \\ \qquad \lambda_{k-1}\varphi(g_{k-1}(p^{k-1})) - \lambda_{k-1}\varphi(p), \forall p \in H, \\ \langle \lambda_k A_k(p^1) + g_k(p^k) - f_k(p^1), p - g_k(p^k)\rangle \geq \lambda_k\varphi(g_k(p^k)) - \lambda_k\varphi(p), \forall p \in H, \\ for\ each\ \lambda_i > 0, i \in \{1,2,\ldots,k\}. \end{cases} \tag{1.5}$$

(II) If φ is an indicator function of $D \subset H$, then GSNMVIP (1.4) reduced to system of general variational inequalities (GSVIP)to find $(p^1, p^2, \ldots p^k) \in \underbrace{D \times D \times \ldots \times D}_{(k\ times)}$ such that

$$\begin{cases} \langle \lambda_1 A_1(p^2, p^3, \ldots p^k, p^1) + g_1(p^1) - f_1(p^2), p - g_1(p^1)\rangle \geq 0, \forall p \in D, \\ \langle \lambda_2 A_2(p^3, p^4, \ldots p^1, p^2) + g_2(p^2) - f_2(p^3), p - g_2(p^2)\rangle \geq 0, \forall p \in D, \\ \vdots \\ \langle \lambda_{k-1} A_{k-1}(p^K, p^1, \ldots p^{k-2}, p^{k-1}) + g_{k-1}(p^{k-1}) - f_{k-1}(p^k), p - g_{k-1}(p^{k-1})\rangle \geq 0, \forall p \in D, \\ \langle \lambda_k A_k(p^1, p^2, \ldots p^{k-1}, p^k) + g_k(p^k) - f_k(p^1), p - g_k(p^k)\rangle \geq 0, \forall p \in D, \\ for\ each\ \lambda_i > 0, i \in \{1,2,\ldots,k\}. \end{cases} \tag{1.6}$$

If $k = 1,2$, and $f_i = g_i = I$, then problem (1.4) reduces to Noor's result discussed in [6]. If $k = 1,2,3$ and φ is an indicator function of D, then problem (1.6) reduce to Zhang's result discussed in [13]. Here we can establish the various results, for example see [2, 3, 4, 5, 7].

2. Explicit iterative algorithms

Lemma (1.1) permit us to write GSNMVIP (1.4) equivalently to fixed point problem as follows:

$$\begin{cases} g_1(p^1) = J_\varphi[f_1(p^2) - \lambda_1 A_1(p^2, p^3, \ldots p^k, p^1)], \\ g_2(p^2) = J_\varphi[f_2(p^3) - \lambda_2 A_2(p^3, p^4, \ldots p^1, p^2)], \\ \vdots \\ g_{k-1}(p^{k-1}) = J_\varphi[f_{k-1}(p^k) - \lambda_{k-1} A_{k-1}(p^k, p^1, \ldots p^{k-2}, p^{k-1})], \\ g_k(p^k) = J_\varphi[f_k(p^1) - \lambda_k A_k(p^1, p^2, \ldots p^{k-1}, p^k)], \\ for\ each\ \lambda_i > 0, i \in \{1,2,\ldots,k\}. \end{cases} \tag{2.1}$$

Here, we put forward the k-steps explicit iterative algorithms to find the approximate solution of GSNMVIP (1.4) by using its alternative equivalent fixed point formulations (2.1).

Algorithm 2.1 *For any* $(p_0^1, p_0^2, \ldots p_0^k) \in \underbrace{H \times H \times \ldots \times H}_{k\ times}$, *compute the sequences* $\{p_n^1\}, \{p_n^2\}, \ldots \{p_n^k\}$ *by*

$$\begin{cases} p_{n+1}^1 = (1-\varepsilon_n)p_n^1 + \varepsilon_n[p_n^1 - g_1(p_n^1) + J_\varphi[f_1(p_n^2) - \lambda_1 A_1(p_n^2, p_n^3, \ldots p_n^k, p_n^1)]], \\ g_2(p_{n+1}^2) = J_\varphi[f_2(p_{n+1}^3) - \lambda_2 A_2(p_{n+1}^3, p_n^4, \ldots p_n^1, p_n^2)], \\ \vdots \\ g_{k-1}(p_{n+1}^{k-1}) = J_\varphi[f_{k-1}(p_{n+1}^k) - \lambda_{k-1} A_{k-1}(p_{n+1}^k, p_n^1, \ldots p_n^{k-2}, p_n^{k-1})], \\ g_k(p_{n+1}^k) = J_\varphi[f_k(p_{n+1}^1) - \lambda_k A_k(p_{n+1}^1, p_n^2, \ldots p_n^{k-1}, p_n^k)], \end{cases} \tag{2.2}$$

where $\lambda_i > 0, i \in \{1,2,\ldots,k\}$ and sequence $\varepsilon_n \in [0,1]$ for all $n \geq 0$.

If $A_i, i \in \{1,2,\ldots,k\}$, is univariate, then Algorithm (2.1) reduces to Algorithm (2.2) as follows:

Algorithm 2.2 *For any* $(p_0^1, p_0^2, \ldots p_0^k) \in \underbrace{H \times H \times \ldots \times H}_{k\ times}$, *compute the sequences* $\{p_n^1\}, \{p_n^2\}, \ldots \{p_n^k\}$ *by*

$$\begin{cases} p_{n+1}^1 = (1-\varepsilon_n)p_n^1 + \varepsilon_n[p_n^1 - g_1(p_n^1) + J_\varphi[f_1(p_n^2) - \lambda_1 A_1(p_n^2)]], \\ g_2(p_{n+1}^2) = J_\varphi[f_2(p_{n+1}^3) - \lambda_2 A_2(p_{n+1}^3)], \\ \vdots \\ g_{k-1}(p_{n+1}^{k-1}) = J_\varphi[f_{k-1}(p_{n+1}^k) - \lambda_{k-1} A_{k-1}(p_{n+1}^k)], \\ g_k(p_{n+1}^k) = J_\varphi[f_k(p_{n+1}^1) - \lambda_k A_k(p_{n+1}^1)], \end{cases} \tag{2.3}$$

where $\lambda_i > 0, i \in \{1,2,\ldots,k\}$ and sequence $\varepsilon_n \in [0,1]$ for all $n \geq 0$.

If φ is an indicator function of $D \subset H$, then Algorithm (2.1) reduces to Algorithm (2.3) as follows:

Algorithm 2.3 *For any* $(p_0^1, p_0^2, \ldots p_0^k) \in \underbrace{D \times D \times \ldots \times D}_{k\ times}$, *compute the sequences* $\{p_n^1\}, \{p_n^2\}, \ldots \{p_n^k\}$ *by*

$$\begin{cases} p_{n+1}^1 = (1-\varepsilon_n)p_n^1 + \varepsilon_n[p_n^1 - g_1(p_n^1) + P_D[f_1(p_n^2) - \lambda_1 A_1(p_n^2, p_n^3, \ldots p_n^k, p_n^1)]], \\ g_2(p_{n+1}^2) = P_D[f_2(p_{n+1}^3) - \lambda_2 A_2(p_{n+1}^3, p_n^4, \ldots p_n^1, p_n^2)], \\ \vdots \\ g_{k-1}(p_{n+1}^{k-1}) = P_D[f_{k-1}(p_{n+1}^k) - \lambda_{k-1} A_{k-1}(p_{n+1}^k, p_n^1, \ldots p_n^{k-2}, p_n^{k-1})], \\ g_k(p_{n+1}^k) = P_D[f_k(p_{n+1}^1) - \lambda_k A_k(p_{n+1}^1, p_n^2, \ldots p_n^{k-1}, p_n^k)], \end{cases} \tag{2.4}$$

where $\lambda_i > 0, i \in \{1,2,\ldots,k\}$ and sequence $\varepsilon_n \in [0,1]$ for all $n \geq 0$.

Definition 2.4 *A mapping* $A: \underbrace{H \times H \times \ldots H}_{k\ times} \to H$

(i) is relaxed (α, β)-cocoercive in the first component if $\exists$ constants $\alpha, \beta > 0$ such that

$$\langle A_1(p^1, p^2, \ldots, p^k) - A_1(p^{1*}, p^{2*}, \ldots, p^{k*}), p^1 - p^{1*} \rangle \geq -\alpha \parallel A_1(p^1, p^2, \ldots, p^k) - A_1(p^{1*}, p^{2*}, \ldots, p^{k*}) \parallel^2 + \beta \parallel p^1 - p^{1*} \parallel^2, \forall p^1, p^{1*} \in H;$$

(ii) is κ-Lipschitz continuous if $\exists$ a constant $\kappa > 0$ such that

$$\| A_1(p^1, p^2, \ldots, p^k) - A_1(p^{1*}, p^{2*}, \ldots, p^{k*}) \| \le \bar{t} \| p^1 - p^{1*} \|, \forall p^1, p^{1*} \in H.$$

Lemma 2.5 *Let us consider nonnegative sequence* $\{p_n\}$, *satisfy* $p_{n+1} \le (1 - \varepsilon_n) p_n + q_n$, $\forall n \ge 0$, *with* $\varepsilon_n \in [0,1]$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, *and* $q_n = o(\varepsilon_n)$. *Then* $\lim_{n\to\infty} p_n = 0$.

3. Convergence Theorem

Theorem 3.1 *Let* $i \in \{1,2,\ldots,k\}$ *and* $(p^{1*}, p^{2*}, \ldots, p^{k*})$ *be the solution of GSNMVIP (1.4). Let* $A_i: \underbrace{H \times H \times \ldots H}_{k\ times} \to H$ *is relaxed* (α_i, β_i)*-cocoercive and* κ_i*-Lipschitzian in the first component. Let* $g_i: H \to H$ *is relaxed* (r_i, s_i)*-cocoercive and* t_i*-Lipschitzian, and* $f_i: H \to H$ *is relaxed* $(\bar{r}_i, \bar{s}_i)$*-cocoercive and* $\bar{t}_i$*-Lipschitzian. If*

$$k_2 < 1, k_3 < 1, \ldots, k_k < 1 \tag{3.1}$$

$$\prod_{i=1}^{k} (1 - k_i) \ge \prod_{i=1}^{k} (k'_i + k''_i), \ where \tag{3.2}$$

$$k_i = [1 + 2r_i t_i^2 - 2s_i + t_i^2]^{1/2}; \tag{3.3}$$

$$k'_i = [1 + 2\lambda_i \alpha_i \kappa_i^2 - 2\lambda_i \beta_i + \lambda_i^2 \kappa_i^2]^{1/2}; \tag{3.4}$$

$$k''_i = [1 + 2\bar{r}_i \bar{t}_i^2 - 2\bar{s}_i + \bar{t}_i^2]^{1/2}; \tag{3.5}$$

and $\varepsilon_n \in [0,1]$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, then iterative sequences $\{p_n^1\}, \{p_n^2\}, \ldots, \{p_n^k\}$ generated by Algorithm (2.1), strongly converges to the solution $(p^{1*}, p^{2*}, \ldots, p^{k*}) \in \underbrace{H \times H \times \ldots \times H}_{k\ times}$.

Proof. Using Algorithm (2.1) and nonexpansive property of the resolvent operator J_φ to evaluate

$$\begin{aligned}
\| p_{n+1}^1 - p^{1*} \| = {} & \| (1 - \varepsilon_n) p_n^1 + \varepsilon_n [p_n^1 - g_1(p_n^1) + J_\phi [f_1(p_n^2) - \lambda_1 A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1)]] \\
& -(1 - \varepsilon_n) p^{1*} - \varepsilon_n [p^{1*} - g_1(p^{1*}) + J_\phi [f_1(p^{2*}) - \lambda_1 A_1(p^{2*}, p^{3*}, \ldots, p^{k*}, p^{1*})]] \| \\
\le {} & (1 - \varepsilon_n) \| p_n^1 - p^{1*} \| + \varepsilon_n \| p_n^1 - p^{1*} - (g_1(p_n^1) - g_1(p^{1*})) \| \\
& + \varepsilon_n \| (p_n^2) - \lambda A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1) - (f_1(p^{2*}) - \lambda_1 A_1(p^{2*}, p^{3*}, \ldots, p^{k*}, p^{1*})) \| \\
\le {} & (1 - \varepsilon_n) \| p_n^1 - p^{1*} \| + \varepsilon_n \| p_n^2 - p^{2*} - \lambda_1 [A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1) \\
& - A_1(p^{2*}, p^{3*}, \ldots, p^{k*}, p^{1*})] \| + \varepsilon_n \| p_n^1 - p^{1*} - (g_1(p_n^1) - g_1(p^{1*})) \| \\
& + \varepsilon_n \| p_n^2 - p^{2*} - (f_1(p_n^2) - f_2(p^{2*}) \|
\end{aligned}$$

. (3.6)

Using relaxed (α_i, β_i)-cocoercivity and κ_i-Lipschitz continuity on A_i in the first component for $i = 1$, we have

$$\begin{aligned}
& \| p_n^2 - p^{2*} - \lambda_1 [A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1) - A_1(p^{2*}, p_n^{3*}, \ldots, p_n^{k*}, p^{1*})] \|^2 \\
& = \| p_n^2 - p^{2*} \|^2 - 2\lambda_1 \langle A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1) - A_1(p^{2*}, p^{3*}, \ldots, p^{k*}, p^{1*}), p_n^2 - p^{2*} \rangle \\
& \quad + \lambda_1^2 \| A_1(p_n^2, p_n^3, \ldots, p_n^k, p_n^1) - A_1(p^{2*}, p^{3*}, \ldots, p^{k*}, p^{1*}) \|^2 \\
& \le [1 + 2\lambda_1 \alpha_1 \kappa_1^2 - 2\lambda_1 \beta_1 + \lambda_1^2 \kappa_1^2] \| p_n^2 - p^{2*} \|^2
\end{aligned}$$

$= k'_1 \parallel p_n^2 - p^{2*} \parallel^2, where\ k'_1\ given\ by$ (3.4) $for\ i = 1.$

(3.7)

Similarly, by relaxed cocoercivity and Lipschitz continuity on g_i and f_i for $i = 1$, we have,

$\parallel p_n^1 - p^{1*} - (g_1(p_n^1) - g_1(p^{1*})) \parallel^2 \leq k_1 \parallel p_n^1 - p^{1*} \parallel$

$, where\ k_1\ given\ by$ (3.3) $for\ i = 1.$ (3.8)

$\parallel p_n^2 - p^{2*} - (f_1(p_n^2) - f_1(p^{2*})) \parallel^2 \leq k''_1 \parallel p_n^2 - p^{2*} \parallel$

$, where\ k''_1\ given\ by$ (3.5) $for\ i = 1.$ (3.9)

Using (3.7)-(3.9) in (3.6), we have

$$\parallel p_{n+1}^1 - p^{1*} \parallel \leq [1 - (1 - k_1)\varepsilon_n] \parallel p_n^1 - p^{1*} \parallel + \varepsilon_n(k'_1 + k''_1) \parallel p_n^2 - p^{2*} \parallel.$$

(3.10)

By relaxed (α_i, β_i)-cocoercivity and κ_i-Lipschitz continuity on A_i in the first component for $i = 2$, we get

$\parallel p_{n+1}^3 - p^{3*} - \lambda_1[A_2(p_{n+1}^3, p_n^4, \ldots, p_n^1, p_n^2) - A_2(p^{3*}, p^{4*}, \ldots, p^{1*}, p^{2*}))] \parallel^2$

$= k'_2 \parallel p_{n+1}^3 - p^{3*} \parallel^2, where\ k'_2$ given by (3.4) for $i =$ 2. (3.11)

Similarly, by relaxed cocoercivity and Lipschitz continuity on g_i and f_i for $i = 2$, we have,

$\parallel p_{n+1}^2 - p^{2*} - (g_2(p_{n+1}^2) - g_2(p^{2*})) \parallel^2 \leq k_2 \parallel p_{n+1}^2 - p^{2*} \parallel$

$, k_2$ given by (3.3) $for\ i = 2.$ (3.12)

$\parallel p_{n+1}^3 - p^{3*} - (f_2(p_{n+1}^3) - f_2(p^{3*})) \parallel^2 \leq k''_2 \parallel p_{n+1}^3 - p^{3*} \parallel$

$, k''_2$ given by (3.5) $for\ i = 2.$ (3.13)

Using Algorithm (2.1) and nonexpansive property of resolvent operator J_φ to evaluate the following

$\parallel g_2(p_{n+1}^2) - g_2(p^{2*}) \parallel$

$= J_\varphi[f_2(p_{n+1}^3) - \lambda_2 A_2(p_{n+1}^3, p_n^4, \ldots, p_n^1, p_n^2)] - J_\varphi[f_2(p^{3*})\lambda_2 - A_2(p^{3*}, p^{4*}, \ldots, p^{1*}, p^{2*})] \parallel$

$\leq \parallel p_{n+1}^3 - p^{3*} - \lambda_2(A_2(p_{n+1}^3, p_n^4, \ldots, p_n^1, p_n^2) - A_2(p^{3*}, p^{4*}, \ldots, p^{1*}, p^{2*})) \parallel$

$+ \parallel p_{n+1}^3 - p^{3*} - (f_2(p_{n+1}^3) - f_2(p^{3*}) \parallel$

$\leq \{k'_2 + k''_2\} \parallel p_{n+1}^3 - p^{3*} \parallel$

(3.14)

Using (3.12) and (3.14), we have

$$\parallel p_{n+1}^2 - p^{2*} \parallel \leq \parallel p_{n+1}^2 - p^{2*} - (g_2(p_{n+1}^2) - g_2(p^{2*})) \parallel + \parallel g_2(p_{n+1}^2) - g_2(p^{2*}) \parallel$$

$$\parallel p_{n+1}^2 - p^{2*} \parallel \leq k_2 \parallel p_{n+1}^2 - p^{2*} \parallel + (k'_2 + k''_2) \parallel p_{n+1}^3 - p^{3*} \parallel$$

$\parallel p_{n+1}^2 - p^{2*} \parallel \leq \frac{(k'_2 + k''_2)}{1 - k_2} \parallel p_{n+1}^3 - p^{3*} \parallel, where\ k_2 < 1\ by$ (3.1), that is

$\parallel p_n^2 - p^{2*} \parallel \leq \frac{(k'_2 + k''_2)}{1 - k_2} \parallel p_n^3 - p^{3*} \parallel, where\ k_2 <$ $1\ by$ (3.1). (3.15)

Through this similar process, we can evaluate

$$\parallel p_n^3 - p^{3*} \parallel \leq \frac{(k'_3 + k''_3)}{1 - k_3} \parallel p_n^4 - p^{4*} \parallel, where\ k_3 < 1 \text{ by (3.1).} \qquad (3.16)$$

:

$$\| p_n^{k-1} - p^{(k-1)*} \| \leq \frac{(k'_{k-1}+k''_{k-1})}{1-k_{k-1}} \| p_n^k - p^{k*} \|, \text{where } k_{k-1} < 1 \text{ by } (3.1), \tag{3.17}$$

$$\| p_n^k - p^{k*} \| \leq \frac{(k'_k+k''_k)}{1-k_k} \| p_n^1 - p^{1*} \|, \text{where } k_k < 1 \text{ by } (3.1). \tag{3.18}$$

Using (3.10), (3.15)-(3.18), we get

$$\| p_{n+1}^1 - p^{1*} \| \leq \left(1 - \varepsilon_n \left(1 - k_1 - (k'_1 + k''_1) \frac{(k'_2 + k''_2)}{1-k_2} \frac{(k'_3 + k''_3)}{1-k_3} \dots \frac{(k'_{k-1} + k''_{k-1})}{1-k_{k-1}} \frac{(k'_k + k''_k)}{1-k_k}\right)\right) \times \| p_n^1 - p^{1*} \|.$$

$$\| p_{n+1}^1 - p^{1*} \| \leq \left(1 - \varepsilon_n \left(1 - k_1 - (k'_1 + k''_1) \prod_{i=2}^k \frac{(k'_i+k''_i)}{1-k_i}\right)\right) \| p_n^1 - p^{1*} \|. \tag{3.19}$$

Since $\left(1 - k_1 - (k'_1 + k''_1) \prod_{i=2}^k \frac{(k'_i+k''_i)}{1-k_i}\right) \in [0,1]$ (3.20)

and $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, from Lemma 2.5, we have $\lim_{n\to\infty} \| p_n^1 - p^{1*} \| = 0$. This completes the proof.

If $k = 1,2$, and $f_i = g_i = I$, then Theorem 3.1 reduces to Noor's result discussed in [6].

Corollary 3.2 *Let $i \in \{1,2,\dots,k\}$ and $(p^{1*}, p^{2*}, \dots, p^{k*})$ be the solution of GSNMVIP (1.5). Assume that univariate mappings $A_i: H \to H$ is relaxed (α_i, β_i)-cocoercive and κ_i-Lipschitzian. Let $g_i: H \to H$ be relaxed (r_i, s_i)-cocoercive and t_i-Lipschitz, and $f_i: H \to H$ be relaxed $(\bar{r}_i, \bar{s}_i)$-cocoercive and $\bar{t}_i$-Lipschitz. If,*

$$k_2 < 1, k_3 < 1, \dots, k_k < 1; \tag{3.21}$$

$$\prod_{i=1}^k (1 - k_i) \geq \prod_{i=1}^k (k'_i + k''_i), \ \textit{where} \tag{3.22}$$

$$k_i = [1 + 2r_i t_i^2 - 2s_i + t_i^2]^{1/2}; \tag{3.23}$$

$$k'_i = [1 + 2\lambda_i \alpha_i \kappa_i^2 - 2\lambda_i \beta_i + \lambda_i^2 \kappa_i^2]^{1/2}; \tag{3.24}$$

$$k''_i = [1 + 2\bar{r}_i \bar{t}_i^2 - 2\bar{s}_i + \bar{t}_i^2]^{1/2}; \tag{3.25}$$

and $\varepsilon_n \in [0,1]$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, then iterative sequences $\{p_n^1\}, \{p_n^2\}, \ldots, \{p_n^k\}$ generated by Algorithm (2.2), strongly converges to the solution $(p^{1*}, p^{2*}, \ldots, p^{k*}) \in \underbrace{H \times H \times \ldots \times H}_{k\ times}$.

Corollary 3.3 *Let $i \in \{1,2,\ldots,k\}$ and $(p^{1*}, p^{2*}, \ldots, p^{k*})$ be the solution of GSVIP (1.6). Assume that $D \subset H$ be a closed convex subset in H and $A_i: \underbrace{H \times H \times \ldots H}_{k\ times} \to H$ is relaxed (α_i, β_i)-cocoercive and κ_i-Lipschitzian in the first component. Let $g_i: H \to H$ be relaxed (r_i, s_i)-cocoercive and t_i-Lipschitz, and $f_i: H \to H$ be relaxed $(\bar{r}_i, \bar{s}_i)$-cocoercive and $\bar{t}_i$-Lipschitz. If,*

$$k_2 < 1, k_3 < 1, \ldots, k_k < 1; \quad (3.26)$$

$$\prod_{i=1}^{k} (1 - k_i) \geq (k'_i + k''_i),\ where \quad (3.27)$$

$$k_i = [1 + 2r_i t_i^2 - 2s_i + t_i^2]^{1/2}; \quad (3.28)$$

$$k'_i = [1 + 2\lambda_i \alpha_i \kappa_i^2 - 2\lambda_i \beta_i + \lambda_i^2 \kappa_i^2]^{1/2}; \quad (3.29)$$

$$k''_i = [1 + 2\bar{r}_i \bar{t}_i^2 - 2\bar{s}_i + \bar{t}_i^2]^{1/2}; \quad (3.30)$$

and $\varepsilon_n \in [0,1]$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, then iterative sequences $\{p_n^1\}, \{p_n^2\}, \ldots, \{p_n^k\}$ generated by Algorithm (2.3), strongly converges to the solution $(p^{1*}, p^{2*}, \ldots, p^{k*}) \in \underbrace{D \times D \times \ldots \times D}_{k\ times}$.

If $k = 1,2,3$, then Corollary 3.3 reduce to Zhang's result discussed in [13]. Here we can give more applications of Theorem 3.1 by giving particular values to k with various condition on f_i, g_i, A_i, for example see [2, 3, 4, 5, 7].

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