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# $\boldsymbol{H}(.,.)$-$\boldsymbol{\varphi}$-$\boldsymbol{\eta}$-Mixed Monotone Mapping with An Application in Semi-Inner Product Space

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**Abstract-**. The objective of this work is to proposed a new notion called $H(.,.)$-$\varphi$-$\eta$-mixed monotone mappings in semi-inner product spaces and define generalized resolvent operator linked with $H(.,.)$-$\varphi$-$\eta$-mixed monotone mappings. Further, we perusal its aspects single-valued property as well as Lipschitz continuity. As an application, we also make an attempt to find the existence of solution of set valued variational inclusion involving nonlinear operators and study the graph convergence of proposed iterative algorithms.

## 1. Introduction.

In 2014, Sahu et al. [13] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [14]. Recently Luo and Huang [10], introduced and studied $(H,\varphi)$-$\eta$-monotone mapping in Banach spaces which provides a unifying framework for various classes of monotone mapping. Most recently, Bhat and Zahoor [1, 2], introduced and studied $(H,\phi)$-$\eta$-monotone mapping in semi-inner product space and discussed the convergence analysis of proposed iterative schemes for some classes of variational inclusion through generalized resolvent operator. For the applications point of view, see [3, 8, 10, 12, 13, 14, 16, 17].

The aim of this work is to investigate the notion $H(.,.)$-$\varphi$-$\eta$-mixed monotone mapping in semi-inner product space. First, we define its resolvent operator and study its characteristics single-valued property as well as Lipschitz continuity. We also make an attempt to find the existence of solution of set valued variational inclusion involving nonlinear operators and study the graph convergence of proposed iterative algorithms. The obtained results are quite similar to above discussed research work but we utilize distinguished notion and approach to solve variational inclusion problems in 2-uniformly smooth Banach space. Our work is the extension and refinement of the existing results, see [1, 2, 5, 6, 8, 10, 17].

For a detailed study and fundamental results on semi-inner product spaces, one may refer to Lumer [11], Giles [4] and Koehler [9].

**Definition 1.1** [13, 15] The Y be a Banach space, then
(i) modulus of smoothness of Y defined as $\varrho_Y(s) = \sup\left\{\frac{\|u^1+v^1\|+\|u^1-v^1\|}{2} - 1: \|u^1\| \leq 1, \|v^1\| \leq s\right\}$; (ii) Y be uniformly smooth if $\lim_{s\to 0} \varrho_Y(s)/s = 0$;
(iii) Y be p-uniformly smooth for $p > 1$, if there exists $c > 0$ such that $\varrho_Y(s) \leq cs^p$;
(iv) Y be 2-uniformly smooth if there exists $c > 0$ such that $\varrho_Y(s) \leq cs^2$.

**Lemma 1.2** [13, 15] *Let $p > 1$ be a real number and $Y$ be a smooth Banach space. Then the following statements are equivalent:*
(i) Y is 2-*uniformly smooth*;
(ii) *There is a constant* $k > 0$ *such that for every* $v^1, w^1 \in Y$, *the inequality holds* $\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2\langle w^1, f_{v^1}\rangle + k\|w^1\|^2$, *where* $f_{v^1} \in J(v^1)$ *and* $J(v^1) = \{v^{1*} \in Y^*: \langle v^1, v^{1*}\rangle = \|v^1\|^2$ *and* $\|v^{1*}\| = \|v^1\|\}$ *is the normalized duality mapping.*

**Remark 1.3** " Every normed linear space Y is a semi-inner product space (see [11]). Infact, by Hahn-Banach theorem, for each $v^1 \in Y$, there exists at least one functional $f_{v^1} \in Y^*$ such that $\langle v^1, f_{v^1}\rangle = \|v^1\|^2$. Given any such mapping $f: Y \to Y^*$, we can verify that $[w^1, v^1] = \langle w^1, f_{v^1}\rangle$ defines a semi-inner product. Hence we have the inequality $\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2[w^1, f_{v^1}] + s\|w^1\|^2$. The constant s is called constant of smoothness of Y, is chosen with best possible minimum value", [13].

## 2. Preliminaries

Let $Y$ be a 2-uniformly smooth Banach space. Its norm and topological dual space is given by $\|.\|$ and $Y^*$, respectively. The semi-inner product $[.,.]$ signify the dual pair among $Y$ and $Y^*$.

In order to proceed the next, we recall some basic concepts, which will be needed in the subsequent sections.

**Definition 2.1** [10, 13] Let $Y$ be real *2*-uniformly smooth Banach space. Let single-valued mappings $H, \eta: Y \times Y \to Y$, and $Q, R: Y \to Y$, then
(i) $H(Q,.)$ is $(\mu, \eta)$-cocoercive in regards R if there $\exists$ constant $\mu > 0$ such that

$$[H(Qu, x) - H(Qu', x), \eta(u, u')] \geq \mu \|Qu - Qu'\|^2, \forall x, u, u' \in Y;$$

(ii) $H(.,R)$ is $(\gamma, \eta)$-relaxed monotone in regards R if there $\exists$ constant $\gamma > 0$ such that

$$[H(x, Ru) - H(x, Ru'), \eta(u, u')] \geq -\gamma \|u - u'\|^2, \forall x, u, u' \in Y;$$

(iii) $H(Q,.)$ is $\kappa_1$-Lipschitz continuous in regards Q if there $\exists$ constant $\kappa_1$ such that

$$\| H(Qu,x) - H(Qu',x) \| \leq \kappa_1 \| u - u' \|, \forall x,u,u' \in Y;$$

(iv) $\eta$ is be τ-Lipschitz continuous if there ∃ constant $\tau > 0$ such that

$$\| \eta(u,u') \| \leq \tau \| u - u' \|, \forall u,u' \in Y.$$

**Definition 2.2** Let $\eta: Y \times Y \to Y$ be the mapping and $M: Y \times Y \multimap Y$ be the multi-valued mapping. Then $M$ is called $(m,\eta)$-relaxed monotone if ∃ a constant $m > 0$ such that

$$[v^* - w^*, \eta(v,w)] \geq -m \| v - w \|^2, \forall v,w \in Y, v^* \in M(v,t), w^* \in M(w,t), for\ each\ fixed\ t \in Y;$$

**Definition 2.3** A multi-valued mapping $S: Y \to CB(Y)$ is called $D$-Lipschitz continuous with constant $\lambda_S > 0$, if

$$D(Sv,Sw) \leq \lambda_S \| v - w \|, \quad \forall v,w \in Y, where\ D(.,.) \text{ is Hausdorff metric } CB(Y).$$

## 3. Preliminaries

Let $Y$ be 2-uniformly smooth Banach space. Assume that $\eta, H: Y \times Y \to Y$, and $\varphi, Q, R: Y \to Y$ be single-valued mappings and $M: Y \times Y \multimap Y$ be a multi-valued mapping.

**Definition 3.1** [6] Let $H(.,.)$ is $(\mu,\eta)$-cocoercive in regards $Q$ with non-negative constant $\mu$ and $(\gamma,\eta)$-relaxed monotone in regards $R$ with non-negative constant $\gamma$, then $M$ is called $H(.,.)$-$\varphi$-$\eta$- *mixed monotone* in regards $Q$ and $R$ if for each fixed $t$, $\varphi o M(.,t)$ is $(m,\eta)$-relaxed monotone in regards first argument and $(H(.,.) + \lambda \varphi o M(.,t))(Y) = Y, \lambda > 0$.

Let us consider the following

**Assumption $M_1$:** Let $H$ is $(\mu,\eta)$-cocoercive in regards $Q$ with non-negative constant $\mu$ and $(\gamma,\eta)$-relaxed monotone in regards $R$ with non-negative constant $\gamma$ with $\mu > \gamma$.

**Assumption $M_2$:** Let $Q$ is $\alpha$-expansive.

**Assumption $M_3$:** Let $\eta$ is $\tau$-Lipschitz continuous.

**Assumption $M_4$:** Let $M$ is $H(.,.)$-$\varphi$-$\eta$-mixed monotone mapping in regards $Q$ and $R$ for each fixed $t \in Y$.

**Theorem 3.2** [6] *Let assumptions $M_1$, $M_2$ and $M_4$ hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$, then $(H(Q,R) + \lambda\varphi o M(.,t))^{-1}$ is single-valued.*

**Definition 3.3** [6] Let assumptions $M_1$, $M_2$ and $M_4$ hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$ then the *resolvent operator* $R_{M(.,t),\varphi}^{H(.,.)-\eta}: Y \to Y$ is given as $R_{M(.,t),\varphi}^{H(.,.)-\eta}(u) = (H(Q,R) + \lambda\varphi o M(.,t))^{-1}(u), \ \forall u \in Y$.

**Theorem 3.4** [6] *Let assumptions $M_1$-$M_4$ hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$ and $\eta$ is $\tau$-Lipschitz then $R_{M(.,t),\varphi}^{H(.,.)-\eta}: Y \to Y$ is $\frac{\tau}{\ell - m\lambda}$-Lipschitz continuous, that is,*

$$\| R^{H(.,.)-\eta}_{M(.,t),\varphi}(y) - R^{H(.,.)-\eta}_{M(.,t),\varphi}(z) \| \leq \frac{\tau}{\ell - m\lambda} \| y - z \|, \forall\, y, z \in Y, and\ fixed\ t \in Y.$$

Here, we are given the graph convergence for $H(.,.)$-$\varphi$-$\eta$-mixed monotone mapping

**Definition 3.5** Let set-valued mappings $\{M^k\}$, $M: Y \multimap Y$ are $H(.,.)$-$\varphi$-$\eta$-mixed monotone mappings in regards $Q$ and $R$ for $k = 0,1,2, \ldots$. The sequence $M^k$ is graph-convergent to $M$, denoted by $M^k \overset{H^{\varphi}_{\eta} G}{\longrightarrow} M$, if for every $(x, y) \in M$ there exists a sequence $\{(x_k, y_k)\} \in graph(M^k)$ such that $x_k \to x, y_k \to y\ \ as\ k \to \infty$.

**Lemma 3.6** *Let set-valued mappings $\{M^k\}$, $M: Y \times Y \multimap Y$ be $H(.,.)$-$\varphi$-$\eta$-mixed monotone mappings on $Y$ for $k = 0,1,2, \ldots$. with assumptions $M_1$-$M_4$ hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$. $H(Q, R)$ is $\kappa_1, \kappa_2$ Lipschtiz continuous in regards first and second component, respectively. Then sequence $M^k \overset{H^{\varphi}_{\eta} G}{\longrightarrow} M$ if and only if $R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}(u) \to R^{H(.,.)-\eta}_{M(.,t),\varphi}(u)$ for all $u \in Y$ and $\lambda > 0$, where*

$$R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi} = (H(.,.) + \lambda\varphi o M^k(.,t_k))^{-1}, R^{H(.,.)-\eta}_{M(.,t),\varphi} = (H(.,.) + \lambda\varphi o M(.,t))^{-1}\ for\ each\ fixed\ t_k, t, respectively.$$

The proof of the above lemma can be easily obtained.

## 4 Formulation of the Problem and Existence of Solution.

Let $Y$ be 2-uniformly smooth Banach space. Let $S, T, G: Y \to CB(Y)$ be the multi-valued mappings, and let $Q, R, \varphi: Y \to Y$, $P: Y \times Y \to Y$ and $\eta, H: Y \times Y \to Y$ be single-valued mappings. Suppose that multi-valued mapping $M: Y \times Y \multimap Y$ be a $H(.,.)$-$\varphi$-$\eta$-mixed monotone mapping in regards $Q, R$. We consider the following generalized set-valued variational like inclusion problem to find $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ such that

$$0 \in P(v, w) + M(u, t). \tag{4.1}$$

Huang et al. [7] studied similar problem to (4.1) when $M$ is maximal monotone operator in Hilbert space.

**Lemma 4.1** *Let us consider the mapping $\varphi: Y \to Y$ such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $Ker(\varphi) = \{0\}$, where $Ker(\varphi) = \{v \in Y: \varphi(v) = 0\}$. If $(u, v, w, t)$, where $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ is a solution of problem (4.1) if and only if $(u, v, w, t)$ satisfies the following relation:*

$$u = R^{H(.,.)-\eta}_{M(.,t),\varphi}[H(Qu, Ru) - \lambda\varphi o P(v, w)]. \tag{4.2}$$

**Theorem 4.2** *Let us consider the problem (4.1) with assumptions $M_1$-$M_3$ hold good and $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $Ker(\varphi) = \{0\}$. Let $S, T$ and $G$ are $\lambda_S, \lambda_T$ and*

*$\lambda_G$ D-Lipschitz continuous, and $H(Q,R)$ is $\kappa_1$, $\kappa_2$-Lipschitz continuous in regards $Q$ and $R$, respectively. Let $\varphi oF$ is $(\nu,\eta)$-relaxed monotone in first component and $\epsilon_1,\epsilon_2$-Lipschitz continuous in the first and second component, respectively with*

$0<\sqrt{\{(\kappa_1+\kappa_2)^2+2\nu\lambda\lambda_S^2+2\epsilon_1\lambda\lambda_S((\kappa_1+\kappa_2)+\tau\lambda_S)+\epsilon_1^2\lambda^2\lambda_S^2\}}<\frac{(1-\xi\lambda_G)(\ell-m\lambda)}{\tau}-\epsilon_2\lambda\lambda_T$ and

$$\| R_{M(.,t)}^{H(.,.)-\varphi-\eta}(u)-R_{M(.,t^*)}^{H(.,.)-\varphi-\eta}(u)\|\leq\xi\| t-t^*\|,\forall\, t,t^*\in Y,\xi>0.$$

Then problem (4.1) has a solution $(u,v,w,t)$.

**Proof:** Let $A:Y\to Y$ be single-valued mapping such that

$$A(u)=\ R_{M(.,t),\varphi}^{H(.,.)-\eta}\,[H(Qu,Ru)-\lambda\varphi oP(v,w)]. \tag{4.3}$$

By Lemma 4.1, it is sufficient to show that the mapping $A$ is a contraction. Since $S,T,G$ are $D$-Lipschitz continuous, then

$$\| v-v^*\|\leq D(S(u),S(u^*))\leq\lambda_S\| u-u^*\|, \tag{4.4}$$

$$\| w-w^*\|\leq D(T(u),T(u^*))\leq\lambda_T\| u-u^*\|, \tag{4.5}$$

$$\| t-t^*\|\leq D(G(u),G(u^*))\leq\lambda_G\| u-u^*\|. \tag{4.6}$$

$$\begin{aligned}\| A(u)-A(u^*)\| &=\\ &\| R_{M(.,t),\varphi}^{H(.,.)-\eta}\,[H(Qu,Ru)-\lambda\varphi oP(v,w)]-R_{M(.,t^*),\varphi}^{H(.,.)-\eta}\,[H(Qu^*,Ru^*)\\ &-\lambda\varphi oP(v^*,w^*)]\,\|\\ &\leq\frac{\tau}{\ell-m\lambda}\\ &\| H(Qu,Ru)-H(Qu^*,Ru^*)-\lambda\big(\varphi oP(v,w)-\varphi oP(v^*,w)\big)\|\\ &\quad+\frac{\tau}{\ell-m\lambda}\lambda\|\varphi oP(v^*,w)-\varphi oP(v^*,w^*)\|+\xi\| t-t^*\|.\end{aligned}$$

(4.7)

$$\begin{aligned}&\| H(Qu,Ru)-H(Qu^*,Ru^*)-\lambda(\varphi oP(v,w)-\varphi oP(v^*,w))\|^2\\ &\leq\| H(Qu,Ru)-H(Qu^*,Ru^*)\|^2\\ &\quad-2\lambda[\varphi oP(v,w)-\varphi oP(v^*,w),\eta(v,v^*)]\\ &\quad+2\lambda\|\varphi oP(v,w)-\varphi oP(v^*,w)\\ &\qquad\|\times\{\| H(Qu,Ru)-H(Qu^*,Ru^*)\|+\|\eta(v,v^*)\|\}\\ &\quad+\lambda^2\| oP(v,w)-\varphi oP(v^*,w)\|^2.\end{aligned}$$

(4.8)

By the $\kappa_1,\kappa_2$-Lipschitz continuity of $H(.,.)$ in the first and second component, respectively. We get

$$\| H(Qu,Ru)-H(Qu^*,Ru^*)\|\leq(\kappa_1+\kappa_2)\| u-u^*\|. \tag{4.9}$$

By $\epsilon_1,\epsilon_2$-Lipschitz continuity and $(\nu,\eta)$-relaxed monotonicity $\varphi oP(.,.)$, and (4.6),(4.7), we have

$$\|\varphi oP(v,w)-\varphi oP(v^*,w))\|\leq\epsilon_1\| v-v^*\|\leq\epsilon_1 D(S(u),S(u^*))\leq\epsilon_1\lambda_S\| u-u^*\|, \tag{4.10}$$

$$\| \varphi oP(v^*, w) - \varphi oP(v^*, w^*)) \| \leq \epsilon_2 \| w - w^* \| \leq \epsilon_2 D(T(u), T(u^*)) \leq \epsilon_2 \lambda_T \| u - u^* \|, \qquad (4.11)$$

$$[\varphi oP(v, w) - \varphi oP(v^*, w), \eta(v, v^*)] \leq -\nu \| v - v^* \|^2 \leq -\nu \lambda_S^2 \| u - u^* \|^2. \qquad (4.12)$$

Using (4.8)-(4.12), we get

$$\| H(Qu, Ru) - H(Qu^*, Ru^*) - \lambda(\varphi oP(v, w) - \varphi oP(v^*, w)) \| \leq \sqrt{\{(\kappa_1 + \kappa_2)^2 + 2\nu\lambda\lambda_S^2 + 2\epsilon_1\lambda\lambda_S((\kappa_1 + \kappa_2) + \tau\lambda_S) + \epsilon_1^2\lambda^2\lambda_S^2\}} \| u - u^* \|. \qquad (4.13)$$

Using (4.6), (4.13) in (4.7), we get

$$\| A(u) - A(u^*) \| \leq \Theta \| u - u^* \|,$$

$where$

$$\Theta = [\frac{\tau}{\ell - m\lambda} [\sqrt{\{(\kappa_1 + \kappa_2)^2 + 2\nu\lambda\lambda_S^2 + 2\epsilon_1\lambda\lambda_S((\kappa_1 + \kappa_2) + \tau\lambda_S) + \epsilon_1^2\lambda^2\lambda_S^2\}} + \epsilon_2\lambda\lambda_T] + \xi\lambda_G].$$

Using the given condition, we have $0 < \Theta < 1$. Hence, by Banach contraction principle, $A$ has a fixed point (say) $u \in Y$. Hence, we get $A(u) = R_{M(.,t),\varphi}^{H(.,.)-\eta}[H(Qu, Ru) - \lambda\varphi oP(v, w)]$.

Lemma 4.1 permit us to suggest the following iterative scheme to find the approximate solution of (4.1).

**Algorithm 4.3** For any given $z_0 \in Y$, we can choose $u_0 \in Y$, $v_0 \in S(u_0)$, $w_0 \in T(u_0)$, $t_0 \in G(u_0)$ and $0 < \epsilon < 1$ such that sequences $\{u_k\}$, $\{v_k\}$, $\{w_k\}$ and $\{t_k\}$ satisfy

$$\begin{cases} u_{k+1} = R_{M^k(.,t),\varphi}^{H(.,.)-\eta}(z_k), \\ v_k \in S(u_k),\ \| v_k - v_{k+1} \| \leq D(S(u_k), S(u_{k+1})) + \epsilon^{k+1} \| u_k - u_{k+1} \|, \\ w_k \in T(u_k), \| w_k - w_{k+1} \| \leq D(T(u_k), T(u_{k+1})) + \epsilon^{k+1} \| u_k - u_{k+1} \|, \\ t_k \in G(u_k),\ \| t_k - t_{k+1} \| \leq D(G(u_k), G(u_{k+1})) + \epsilon^{k+1} \| u_k - u_{k+1} \|, \\ z_{k+1} = H(Qu_k, Ru_k) - \lambda\varphi oP(v_k, w_k), \end{cases}$$

where $\lambda > 0$, $k \geq 0$, and $D(.,.)$ is the Hausdorff metric on $CB(Y)$.

**Theorem 4.4** *Let us consider the problem (4.1) with assumptions $M_1$-$M_4$. Let $M^k: Y \times Y \multimap Y$ be $H(.,.)$-$\varphi$-$\eta$ mixed monotone such that $M^k \xrightarrow{H_\eta^\varphi G} M$ as $k \to \infty$. let $\varphi: Y \to Y$ be a single-valued mapping with $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $Ker(\varphi) = \{0\}$. Let $S, T$ and $G$ are $\lambda_S, \lambda_T$ and $\lambda_G$ D-Lipschitz continuous and $H(Q, R)$ is $\kappa_1$, $\kappa_2$-Lipschitz continuous in regards $A$ and $B$, respectively. Let $\varphi oF$ is $(\nu, \eta)$-relaxed monotone in the first component and $\epsilon_1, \epsilon_2$-Lipschitz continuous in the first and second component, respectively with*

$$0 < \sqrt{\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\lambda_S^2 + 2\epsilon_1\lambda\lambda_S((\kappa_1+\kappa_2)+\tau\lambda_S) + \epsilon_1^2\lambda^2\lambda_S^2\}}$$
$$< \frac{(1-\xi\lambda_G)(\ell - m\lambda)}{\tau} - \epsilon_2\lambda\lambda_T$$

and $\| R^{H(.,.)-\varphi-\eta}_{M^k(.,t_k)}(u) - R^{H(.,.)-\varphi-\eta}_{M^{k-1}(.,t_{k-1})}(u) \| \leq \xi \| t_k - t_{k-1} \|, \forall\, t, t^* \in Y, \xi > 0.$

Then the iterative sequences $\{u_k\}, \{v_k\}, \{w_k\}$, and $\{t_k\}$ generated by Algorithm 4.3 converges strongly to the unique solution $(u, v, w, t)$ of SGVI (4.1).

**Proof.** Using Algorithms 4.3 and $\lambda_S, \lambda_T, \lambda_G$-$D$ Lipschitz continuity of $S$, $T$ and $G$, we have

$\| v_k - v_{k-1} \| \leq D(S(u_k), S(u_{k+1})) + \epsilon^k \| u_k - u_{k+1} \| \leq \{\lambda_S + \epsilon^k\} \| u_k - u_{k+1} \|, k = 1, 2, \ldots.$ (4.14)

$\| w_k - w_{k-1} \| \leq D(T((u_k), T(u_{k-1})) + \epsilon^k \| u_k - u_{k+1} \| \leq \{\lambda_T + \epsilon^k\} \| u_k - u_{k+1} \|\ \ k = 1, 2, \ldots.,$ (4.15)

$\| t_k - t_{k-1} \| \leq D(G((u_k), G(u_{k-1})) + \epsilon^k \| u_k - u_{k+1} \| \leq \{\lambda_G + \epsilon^k\} \| u_k - u_{k+1} \|, k = 1, 2, \ldots..$ (4.16)

By Lipschitz continuity of resolvent operator and second condition, we have

$$\begin{aligned}
\| u_{k+1} - u_k \| \leq\ & \| R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}[H(Qu_k, Ru_k) - \lambda\varphi o P(v_k, w_k)] \\
& - R^{H(.,.)-\eta}_{M^{k-1}(.,t_{k-1}),\varphi}[H(Qu_{k-1}, Ru_{k-1}) - \lambda\varphi o P(v_{k-1}, w_{k-1})] \| \\
\leq & \| R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}[H(Qu_k, Ru_k) - \lambda\varphi o P(v_k, w_k)] \\
& - R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}[H(Qu_{k-1}, Ru_{k-1})\ \ \lambda\varphi o P(v_{k-1}, w_{k-1})] \| \\
+ & \| R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}[H(Qu_{k-1}, Ru_{k-1}) - \lambda\varphi o P(v_{k-1}, w_{k-1})] \\
& - R^{H(.,.)-\eta}_{M^{k-1}(.,t_{k-1}),\varphi}[H(Qu_{k-1}, Ru_{k-1}) - \lambda\varphi o P(v_{k-1}, w_{k-1})] \| \\
\leq & \frac{\tau}{\ell - m\lambda} \| H(Qu_k, Ru_k) - H(Qu_{k-1}, Qk_{k-1}) - \lambda(\varphi o P(v_k, k_k) \\
& - \varphi o P(v_{k-1}, w_k)) \| \\
& + \frac{\tau\lambda}{\ell - m\lambda} \| \varphi o P(v_{k-1}, w_k) - \varphi o P(v_{k-1}, w_{k-1}) \| + \xi \| t_k - t_{k-1} \|.
\end{aligned}$$

(4.17)

In the light of (4.13), we can obtained

$$\| H(Qu_k, Ru_k - H(Qu_{k-1}, Ru_{k-1}) - (\varphi o P(v_k, w_k) - \varphi o P(v_{k-1}, w_k)) \|$$
$$\leq \sqrt{[(\kappa_1+\kappa_2)^2 + 2\nu\lambda\{\lambda_S+\epsilon^k\}^2 + 2\epsilon_1\lambda\{\lambda_S+\epsilon^k\}\{(\kappa_1+\kappa_2)+\tau\{\lambda_S+\epsilon^k\}\} + \epsilon_1^2\lambda^2\{\lambda_S+\epsilon^k\}^2]}$$
$$\| u_k - u_{k-1} \|. \tag{4.18}$$

Thus equation (4.17) becomes

$$\| u_{k+1} - u_k \| \leq \Theta(\epsilon^k) \| u_k - u_{k-1} \|, \tag{4.19}$$

$where \qquad \Theta(\epsilon^k) =$

$$[\frac{\tau}{\ell - m\lambda}[\sqrt{\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\{\lambda_S+\epsilon^k\}^2 + 2\epsilon_1\lambda\{\lambda_S+\epsilon^k\}((\kappa_1+\kappa_2)+\tau\{\lambda_S+\epsilon^k\}) + \epsilon_1^2\lambda^2\{\lambda_S}$$

$$+\epsilon_2\lambda\{\lambda_T+\epsilon^k\}] + \xi\{\lambda_S+\epsilon^k\}].$$

Since $0 < \epsilon < 1$, this implies that $\Theta(\epsilon^k) \to \Theta$ as $k \to \infty$, where

$$\Theta = [\frac{\tau}{\ell - m\lambda}[\sqrt{\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\lambda_S^2 + 2\epsilon_1\lambda\lambda_S((\kappa_1+\kappa_2)+\tau\lambda_S) + \epsilon_1^2\lambda^2\lambda_S\}}$$

$$+ \epsilon_2\lambda\lambda_T] + \xi\lambda_S].$$

It is given that $\Theta < 1$, then $\{u_k\}$ is a Cauchy sequence in Banach space $Y$, then $u_k \to u$ as $k \to \infty$.

From equation (4.14)-(4.16) and Algorithm 4.3, the sequences $\{v_k\}$, $\{w_k\}$ and $\{t_k\}$ are also Cauchy sequences in $Y$. Thus, there exist $v$, $w$ and $t$ such that $v_k \to v$, $w_k \to w$ and $t_k \to t$ as $k \to \infty$. Next we will prove that $v \in S(u)$. Since $v_k \in S(u_k)$, then

$$\begin{aligned} d(v, S(u)) &\leq \| v - v_k \| + d(v_k, S(u)) \\ &\leq \| v - v_k \| + D(S(u_k), S(u)) \\ &\leq \| v - v_k \| + \lambda_S \| u_k - u \| \to 0, \text{as } k \to \infty, \end{aligned}$$

which gives $d(v, S(u)) = 0$. Due to $S(u) \in CB(Y)$, we have $v \in S(u)$. In the same manner, we easily show that $w \in T(u)$ and $t \in G(u)$. By the continuity of $R_{M(.,t),\varphi}^{H(.,.)-\eta}, Q, R, S, T\ G, \varphi o P, \eta\ and\ M$ and Algorithms 4.3, we know that $u, v, w\ and\ t$ satisfy $u = R_{M(.,t),\varphi}^{H(.,.)-\eta}[H(Qu, Ru) - \lambda\varphi o P(v, w)]$. Now using Lemma 4.1, $(u, v, w, t)$ is a solution of the problem (4.1).

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