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THE NUMERICAL SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS BY EULER POLYNOMIALS WITH LEAST–SQUARES METHOD

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ABSTRACT

This study introduced a new technique based on the combination of the least-squares method (LSM) with Euler polynomials for finding the approximate solutions of integro-differential equations (IDEs) subject to the mixed conditions. Three examples of first and second-orders linear Fredholm IDEs (FIDEs) and Volterra IDEs (VIDEs) are considered to illustrate the proposed method. The numerical results comprised to demonstrate the validity and applicability of this method comparisons with the exact solution shown that the competence and accuracy of the present technique.

INTRODUCTION

Presented a method for solving high-order Linear FIDE equations under the mixed conditions in terms of Legendre polynomials under mixed conditions. The method used is the Legendre collocation matrix method and then converting the equation and conditions into matrix equations, which correspond to systems of linear algebraic equations with Legendre coefficients (Yalçınbaş, 2009). Used Cauchy kernel with airfoil polynomials of the first kind, and the numerical solution for some of the integro-differential equations gets a system of linear algebraic equations. The convergence of the method gives some sufficient conditions (Mennouni and Guedjiba, 2010). Studied Bessel polynomials to find approximate solutions of high-order linear VIDEs under the mixed conditions, based on collocation points, practical matrix method, the accuracy and efficiency of the method are proven (Yüzbaşı et al., 2011). The Euler polynomial was used to solve the VIDEs of the pantograph delay type in approximation of the solution. The method is discussed in more detail and compared through numerical examples (Mirzaee et al., 2016). Some of the numerical methods is developed for 2^{nd} order VIDEs by using a Legendre spectrum approach. Provide a rigorous error analysis for the proposed methods, shown that the numerical errors decay exponentially in the L^{∞}-norm and L^2-norm. Numerical examples illustrate the convergence and effectiveness of the numerical methods (Wei and Chen, 2011). Applying a moving least squares method and Chebyshev polynomials for solution of VFIDEs of the second kind. The main advantage of this method it does not need a mesh (Yuksel et al, 2012). Presented Chebyshev polynomials under the mixed conditions of method for solving high order linear VFIDEs. The method depends on the approximation the truncated Chebyshev series. The conditions are transformed into the matrix equations, which match system of linear algebraic equations with the unknown Chebyshev coefficients, and then solving the system yields the Chebyshev coefficients of the solution function (Laeli and Maalek , 2012).A new method based on the Laplace Adomian decomposition with the Bernstein polynomial to solve the VIE and IDE of the first and second types, and through examples and comparison accurate and approximate solutions the method is adopted (Rani and Mishra, 2019). Solved some FIDEs with functional arguments, using a Laguerre collocation method, convert it into a matrix equation that corresponds to a system of linear algebraic equations. The efficiency of the proposed method is then proven through examples (Gurbuz et al., 2013).

$$u^{(k)}(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt \,, c \le x,t \le d, \qquad (1.1)$$

under the mixed conditions

$$\sum_{i=0}^{N-1} \left(c_{ji}u^{(i)}(c) + d_{ji}u^{(i)}(d)\right) = \beta_j \qquad j = 0,1,\dots,N-1 \qquad (1.2)$$

Implementation of Euler Polynomials-Least-Square Method for Solving Integro-Differential Equations

In this section, we implement a new approach based on the Euler polynomials as a basis function combining with LSM to solve the equations (1.1) and (1.2)

Euler Polynomials and Their Properties:

We will define Euler polynomials by the following equation (Cheon, 2013),

$$E_i(x) = \frac{1}{i+1}\sum_{j=1}^{i+1}(2 - 2^{j+1})\binom{i+1}{j} B_j x^{i+1-j} \qquad (2.1)$$

where $B_j = B_j(0)$ is the Bernoulli number for each $j = 0,1,\dots,i$

we assume the approximate solution as

$$u(x) = u_m(x) = \sum_{i=0}^{m} a_i E_i(x) \qquad c \le x \le d \tag{2.2}$$

where a_i are unknown constants and $E_i(x)$ are the Euler polynomial of degrees (i). Substituting equation (2.2) into equation (1.1), we get

$$\sum_{i=0}^{m} a_i E_i{}^{(k)}(x) = f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt, \tag{2.3}$$

Combine of Euler Polynomials with Least- Squares Method

The residual equation has been given by

$$R(x, a_i) = R(x, u_m(x)) = \sum_{i=0}^{m} a_i E_i{}^{(k)}(x) - \left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt \right\} \tag{2.4}$$

Let

$$S(a_0, a_1, \dots, a_m) = \int_c^d [R(x, a_i)]^2 w(x) dx, \tag{2.5}$$

where w(x) is the positive weight function defined in the interval [c, d]. For simplicity set
w (x)=1, thus,

$$S(a_0, a_1, \dots, a_m) = \int_c^d \left[- \left\{ f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt \right\} \right]^2 dx, \tag{2.6}$$

We can get the values of a_i , $i \ge 0$ by minimizing the value of S as follows:

$$\frac{\partial S}{\partial a_i} = 0, i = 0,1, \dots, m \tag{2.7}$$

Then from (2.6) by applying (2.7) get:

$$\frac{\partial S}{\partial a_i} = \int_c^d \left[\sum_{i=0}^{m} a_i E_i^{(k)}(x) - \left\{f(x) + \lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt\right\}\right] dx \times$$
$$\int_c^d \left[E_i^{(k)}(x) - \left\{\lambda_1 \int_0^x k_1(x,t) E_i(t) dt + \lambda_2 \int_c^d k_2(x,t) E_i(t) dt\right\}\right] dx = 0\,, \tag{2.8}$$

thus, (2.8) are generated (m+1) algebraic system of equations in (m+1) unknown
$a_i, i = 0, \cdots, m$, or in the matrix form as follow:

$$W = \begin{pmatrix} \int_c^d R(x,a_0)h_0 dx & \int_c^d R(x,a_1)h_0 dx & \dots & \int_c^d R(x,a_m)h_0 dx \\ \int_c^d R(x,a_0)h_1 dx & \int_c^d R(x,a_1)h_1 dx & \dots & \int_c^d R(x,a_m)h_1 dx \\ \int_c^d R(x,a_0)h_m dx & \int_c^d R(x,a_1)h_m dx & \dots & \int_c^d R(x,a_m)h_m dx \end{pmatrix} \tag{2.9}$$

$$G = \begin{pmatrix} \int_c^d \{f(x)\}h_0 dx & \int_c^d \{f(x)\}h_1 dx & \vdots & \int_c^d \{f(x)\}h_m dx \end{pmatrix} \tag{2.10}$$

where

$$h_i = E_i^{(k)} - \left\{\lambda_1 \int_0^x k_1(x,t) E_i dt + \lambda_2 \int_c^d k_2(x,t) E_i dt dx\right\} \tag{2.11}$$

$$R(x,a_i) = \sum_{i=0}^{m} a_i E_i^{(k)}(x) - \left\{\lambda_1 \int_0^x k_1(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{m} a_i E_i(t)\, dt\right\}, \tag{2.12}$$

$$\mathrm{WA} = \mathrm{G} \text{ or } \mathrm{A} = [\mathrm{W}; \mathrm{G}]. \tag{2.13}$$

Property (Zuppa, 2003) : $\forall x \in \underline{\Omega}$ the matrix W defined in (2.13) is non-singular. □
The equation (1.1) corresponds to a system of $(m+1)$ linear algebraic equations with the unknown Euler coefficients $a_i, i = 0,1, \dots, m$,
Another form of (2.13) by applying the conditions can be explained as

$$[U_i : \beta_i] \quad , i = 0, 1, \dots, N-1$$

where

$$U_i = [u_{i0} \quad u_{i1} \quad u_{i2} \quad \dots \quad u_{iN}] , i = 0, 1, 2 \dots N-1$$

(2.14)
The solution of (1.1) under conditions (1.2) can be get by changing the row matrices (2.14) by the last (m) rows of the matrix form (2.13) we get the new augmented matrix (Sezer and Gulsu, 2005; Kurt and Sezer, 2008; Baykuş and Sezer, 2009; Yalçınbaş, 2009; Yüzbaşı and Sezer, 2011; Yüzbaşı et al., 2011).

$$[\widetilde{W}; \widetilde{G}]$$
$$= \Bigg(\int_c^d R(x,a_0)h_0 dx \int_c^d R(x,a_1)h_0 dx \dots \int_c^d R(x,a_m)h_0 dx \ ;$$
$$G_0 \int_c^d R(x,a_0)h_1 dx \int_c^d R(x,a_1)h_1 dx \dots \int_c^d R(x,a_m)h_1 dx \ ; G_1$$
$$\int_c^d R(x,a_0)h_{m_{N_0}} dx \int_c^d R(x,a_1)h_{m_{N_1}} dx \dots \int_c^d R(x,a_m)h_{m_{N_m}} dx$$
$$; \quad G_{m-N} \quad u_{00} \quad u_{01} \quad u_{0N} \ ; \ \beta_0$$
$$\vdots \quad u_{(m-1)0} \quad u_{(m-1)1} \quad \dots \quad u_{(m-1)N} \ ; \ \beta_{N-1} \Bigg), \qquad (2.15)$$

$$A = \widetilde{W}^{-1}\widetilde{G},$$

therefore, the matrix A is uniquely determined. Also, the equation (1.1) with conditions (1.2) has a unique solution (Al-Humedi, 2020).

CONVERGENCE ANALYSIS

Now we will review an estimate of the errors above based on the numerical methods which introduced in the second section want to prove that as $m \to \infty$ the approximate solution u_m(x) will be converge to the exact solution $u(x)$ of (1.1).

Theorem

(Nadir and Dilmi, 2017) : Let $A: C(\Omega) \to C(\Omega)$ be compact operator where $\Omega \in [c,d]$, and the equation

$$u - Au = f \qquad (3.1)$$

admit a unique solution. Assume that the projections $P_m: C(\Omega) \to V_m(\Omega)$ satisfy to
$\|Q_m A - A\| \to 0$ as $m \to \infty$. Then, for sufficiently large m, the approximate equation

$$u_m - Q_m A u_m = Q_m f \qquad (3.2)$$

has a unique solution for all $f \in C(\Omega)$ and there holds an error estimate

$$\|u - u_m\| \leq m\|u - Q_m u\| \qquad (3.3)$$

with some positive constant m depending A.

ILLUSTRATIVE EXAMPLES

In this section, three numerical examples are performed to check the accuracy and efficiency of the combination of LSM for solving high-orders linear IDEs with Euler polynomials as the basis functions we present some numerical examples then we compare the results of our method with the results of some other methods in (AL-Juburee, 2010; Bildik et al, 2010; Yüzbaşı et al, 2011).

The examples are solved to explain them precisely and the time of accomplishment of the method. The absolute error has been defined
Error $=|u(x)-u_m(x)| \qquad c \le x \le d \;\; , m = 1,2,\ldots..$
where $u(x)$ is the exact solution and $u_m(x)$ is the approximate solution .

Example 1

(AL-Juburee, 2010): we considered the following linear FIDE second kind

$$u'(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt \,, c \le x, t \le d,$$

$$u(0) = 0$$

The exact solution is given as $u(x) = xe^x$.
Where, $f(x) = xe^x + e^x - x \,, k_1(x,t) = 0, k_2(x,t) = x \,, \lambda_1 = 0, \lambda_2 = 1, c = 0, d = 1.$

Solution:

an approximate solution $u(x)$ will be applied using the combination of least-squares with the Euler polynomial defined in the form

$$u(x) = \sum_{i=0}^{m} a_i E_i(x) \;\; ,$$

if $m = 2$.
$R_0 = -x, \;\; R_1 = 1 \;,\;\; R_2 = \frac{13x}{6} - 1, \qquad G = [\,-1.3849 \quad 2.2183 \quad 0.7824\;\,]',$
$W = \begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & -\frac{2}{9} & -\frac{1}{2} & 1 & \frac{1}{12} & -\frac{2}{9} & \frac{1}{12} & \frac{34}{108} \end{pmatrix},$
For the given conditions $u(0) = 1$, the augmented matrices are obtained respectively, as

$$U_0 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \end{bmatrix},$$

If we replace the last first rows of the matrices $W \; and \; G$ by the values of U_0, in, then
$\tilde{G} = [\,-1.3849 \quad 2.2183 \quad 0\;\,]', \qquad \tilde{W} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & -\frac{2}{9} & -\frac{1}{2} & 1 & \frac{1}{12} & 1 & -\frac{1}{2} & 0 \end{pmatrix},$
Thus, the Euler coefficients are calculated as
$A = \tilde{W}^{-1}\tilde{G} = [\,1.3591 \quad 2.7183 \quad 2.1548\;\,]',$
Therefore, the approximate solution of the problem taking $m = 2$ is the exact solution under the given conditions as follows:

$$u_2(x) = -5.0000e - 05 + 0.5635x + 2.1548x^2.$$

Example 2

(Batool and Ahmad, 2017): Considered the following linear VIDE second kind.

$$u''(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt\,, c \le x, t \le d,$$

$$u(0) = 0, u'(0) = 1$$

Where, $f(x) = x\,, k_1(x,t) = (x-t),, k_2(x,t) = 0\,, \lambda_1 = 1, \lambda_2 = 0,$

Solution:

an approximate solution $u(x)$ will be applied using the combination of least-squares with the Euler polynomial defined in the form

$$u(x) = \sum_{i=0}^{m} a_i E_i(x) \ ,$$

if $m = 5$.

$$R_0 = -\frac{x^2}{2}, R_1 = -\frac{x^2(2x\,-\,3)}{12}, R_2 = 2\,-\frac{x^3(x\,-\,2)}{12}, R_3 = 6x\,-\frac{x^2(2x^3 -\,5x^2 +\,5)}{40} -\,3$$

$R_4 = 12x^2 - \frac{x^3(x^3 - 3x^2 + 5)}{30} -\,12x,$

$R_5 = 20x^3 -\,30x^2 - \frac{x^2(4x^5 - 14x^4 + 35x^2 - 42)}{168} +\,5$

$$G = [\,-\frac{1}{8}\quad \frac{7}{240}\quad \frac{367}{360}\quad \frac{1621}{3360}\quad -\frac{573}{560}\quad -\frac{5833}{6048}]'$$

$W = (0.0500\ \ -0.0111\ \ -0.3413\ \ -0.2433\ \ 0.3095\ \ 0.4864\ -$
$0.0111\ \ 0.0026\ \ 0.0851\ \ 0.0484\ \ -0.0854\ \ -0.0980\ -$
$0.3413\ \ 0.0851\ \ 4.1013\ \ -0.0094\ \ -4.1039\ \ 0.0200\ -$
$0.2433\ \ 0.0484\ \ -0.0094\ \ 2.9402\ \ 0.0013\ -$
$5.8789\ 0.3095\ \ -0.0854\ \ -4.1039\ \ 0.0013\ \ 4.9034\ -$
$0.0037\ 0.4864\ \ -0.0980\ \ 0.0200\ \ -5.8789\ \ -0.0037\ \ 11.8974\,),$

for the given conditions then augmented matrices are obtained respectively, as follows

$U_0 = [\,1\ \ -\frac{1}{2}\ \ 0\ \ \frac{1}{4}\ \ 0\ \ -\frac{1}{2}\ \ ]\ ,\ U_1 = [\,0\ \ 1\ \ -1\ \ 0\ \ 1\ \ 0\ \ ]$

if we replace the last two rows of the matrices W and G by the values of U_0 and U_1 in, then

$$\tilde{G} = [\ \ -\frac{1}{8}\quad \frac{7}{240}\quad \frac{367}{360}\quad \frac{1621}{3360}\quad 0\quad 1]'$$

$\widetilde{W} = (0.0500\ \ -0.0111\ \ -0.3413\ \ -0.2433\ \ 0.3095\ \ 0.4864\ -$
$0.0111\ \ 0.0026\ \ 0.0851\ \ 0.0484\ \ -0.0854\ \ -0.0980\ -$
$0.3413\ \ 0.0851\ \ 4.1013\ \ -0.0094\ \ -4.1039\ \ 0.0200\ -$
$0.2433\ \ 0.0484\ \ -0.0094\ \ 2.9402\ \ 0.0013\ -$
$5.8789\ 1.0000\ \ -0.5000\ \ 0\ \ 0.2500\ \ 0\ -$
$0.5000\ 0\ \ 1.0000\ \ -1.0000\ \ 0\ \ 1.0000\ \ 0\,)$,

thus, Euler coefficients are calculated as

$$A = \widetilde{W}^{-1}\tilde{G}$$
$$= [\, 0.5876 \quad 1.2715 \quad 0.2936 \quad 0.2117 \quad 0.0221 \quad 0.0095 \,]'$$

Therefore, the approximate solution of the problem taking $m = 5$ is the exact solution under the given conditions as follows:

$$u_5 = 2.5000e - 05 + x - 2.0000e - 04x^2 + 0.1675x^3 - 0.0016x^4 + 0.0095x^5$$

Example 3

(Bildik et al, 2010): Consider the linear FIDEs equation.

$$u''(x) + Au'(x) + Bu(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)u(t)dt + \lambda_2 \int_c^d k_2(x,t)u(t)dt \,, c \le x, t \le d,$$

$$u(0) = u'(0) = 1$$

Where, $A = x, B = -x, f(x) = e^x - 2sinx \,, k_1(x,t) = 0,, k_2(x,t) = sinxe^{-t} \,, \lambda_1 = 0, \lambda_2 = 1,$

$$c = -1, d = 1$$

Solution:

an approximate solution $u(x)$ will be applied using the combination of least-squares with the Euler polynomial which defined in the form

$$u(x) = \sum_{i=0}^{m} a_i E_i(x) \;,$$

if $m = 6$.

$$R_0 = -x - 2sinh(1)sin(x),$$

$$R_1 = x - x\left(x - \frac{1}{2}\right) + \frac{e^{-1}sin(x)(e^2 + 3)}{2},$$

$$R_2 = x(2x - 1) + x(-x^2 + x) - e^{-1}sin(x)(e^2 - 3) + 2,$$

$$R_3 = 6x - x(-3x^2 + 3x) - x\left(x^3 - \frac{3x^2}{2} + \frac{1}{4}\right) - \frac{e^{-1}sin(x)(3e^2 - 35)}{4} - 3$$

$$R_4 = x(4x^3 - 6x^2 + 1) - 12x - x(x^4 - 2x^3 + x) + 12x^2 - 5e^{-1}sin(x)(e^2 - 7),$$

$$R_5 = x(5x^4 - 10x^3 + 5x) - x\left(x^5 - \frac{5x^4}{2} + \frac{5x^2}{2} - \frac{1}{2}\right) - 30x^2 + 20x^3 - \frac{e^{-1}sin(x)(47e^2 - 351)}{2} + 5,$$

$$R_6 = 30x + x(-x^6 + 3x^5 - 5x^3 + 3x) - 60x^3 + 30x^4 + x(6x^5 - 15x^4 + 15x^2 - 3) - 13e^{-1}sin(x)(11e^2 - 81)$$

$$G = [\,1.4730 \quad -2.3986 \quad 8.7551 \quad -14.6057 \quad 20.1751 \quad -22.5255 \quad 22.4209]'$$

$$W = \begin{pmatrix} 6.5109 & -6.7241 & 6.3571 & -19.7438 & 32.5235 & -30.4500 & 15.5392 \\ -6.7241 & 7.3446 & -9.1028 & 23.8964 & -39.7359 & 41.2955 & -26.6269 \\ 6.3571 & -9.1028 & 25.8631 & -46.8102 & 70.0132 & -82.2648 & 78.5555 \\ -19.7438 & 23.8964 & -46.8102 & 99.1612 & -152.8460 & 168.8107 & -145.6856 \\ 32.5235 & -39.7359 & 70.0132 & -152.8460 & 260.1630 & -315.7315 & 265.4038 \\ -30.4500 & 41.2955 & -82.2648 & 168.8107 & -315.7315 & 447.5266 & -425.4950 \\ 15.5392 & -26.6269 & 78.5555 & -145.6856 & 265.4038 & -425.4950 & 513.4506 \end{pmatrix}$$

from the given conditions the augmented matrices are obtained respectively, as follows

$U_0 = [\,1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{4} \quad 0 \quad -\frac{1}{2} \quad 0\,]$, $U_1 = [\,0 \quad 1 \quad -1 \quad 0 \quad 1 \quad 0 \quad -3]$

if we replace the last two rows of the matrices W and G by the values of U_0 and U_1 in, then

$$\tilde{G} = [\,1.4730 \quad -2.3986 \quad 8.7551 \quad -14.6057 \quad 20.1751 \quad 1 \quad 1]'$$

$$\widetilde{W} = \begin{pmatrix} 6.5109 & -6.7241 & 6.3571 & -19.7438 & 32.5235 & -30.4500 & 15.5392 \\ -6.7241 & 7.3446 & -9.1028 & 23.8964 & -39.7359 & 41.2955 & -26.6269 \\ 6.3571 & -9.1028 & 25.8631 & -46.8102 & 70.0132 & -82.2648 & 78.5555 \\ -19.7438 & 23.8964 & -46.8102 & 99.1612 & -152.8460 & 168.8107 & -145.6856 \\ 32.5235 & -39.7359 & 70.0132 & -152.8460 & 260.1630 & -315.7315 & 265.4038 \\ 1.0000 & -0.5000 & 0 & 0.2500 & 0 & -0.5000 & 0 \\ 0 & 1.0000 & -1.0000 & 0 & 1.0000 & 0 & -3.0000 \end{pmatrix}$$

$$A = \widetilde{W}^{-1}\tilde{G} = [\,1.8591 \quad 1.8591 \quad 0.9292 \quad 0.3080 \quad 0.0744 \quad 0.0131 \quad 0.0014\,]'$$

Therefore, the approximate solution of the problem taking $m = 6$ is the exact solution under the given conditions as follows:

$$u_6 = 1 + 1.0001x + 0.5000x^2 + 0.1662x^3 + 0.0416x^4 + 0.0089x^5 + 0.0014x^6$$

Table 1. Exact, approximate solutions and the errors with m=2 for Example 1.

x	Exact Solution	Approximated Solutions	$\lvert u - \tilde{u}\rvert$	Method (AL-Juburee, 2010)

0	0	-5.0000e-05	5.0000e-05	0
0.1	0.1105	0.0778	0.0327	0.0460
0.2	0.2443	0.198842	0.04558	0.0598
0.3	0.4050	0.362932	0.042068	0.0451
0.4	0.5967	0.570118	0.026582	0.0060
0.5	0.8244	0.8204	0.004	0.0526
0.6	1.0933	1.113778	0.020478	0.1254
0.7	1.4096	1.450252	0.040652	0.2063
0.8	1.7804	1.829822	0.049422	0.2881
0.9	2.2136	2.252488	0.038888	0.3629
1	2.7183	2.71825	3.1828e-05	0.4218

Table 2. Exact, approximate solutions and the errors with m=5 for Example 2.

x	Exact Solution	Approximated Solution	$\|u-\tilde{u}\|$	Method (Batool and Ahmad, 2017)
0	0	0.000025	0.2500e-4	0
0.1	0.1001668	0.100190435	0.2364e-4	0.616e-4
0.2	0.2013360	0.20135748	0.2148e-4	0.9827e-4
0.3	0.3045203	0.304539625	0.1932e-4	0.2121e-4
0.4	0.4107523	0.41076932	0.1702e-4	0.15231e-3
0.5	0.5210953	0.521109375	0.1408e-4	0.508e-4

Table 3. Exact, approximate solutions and the errors with m=6 for Example 3

x	Exact Solution	Approximated Solution	$\|u-\tilde{u}\|$	Method (Bildik et al, 2010)
-0.1	0.3678794	0.3678	7.9441e-05	1.95780e-3
-0.8	0.4493289	0.4493156096	1.3355e-05	9.9380e-4
-0.6	0.5488116	0.5488054144	6.2217e-06	6.6730e-4
-0.4	0.6703200	0.6703027584	1.7288e-05	6.6437e-4
-0.2	0.8187307	0.8187142016	1.6551e-05	9.457e-5
0	1.0000	1.0000	0	0
0.2	1.2214027	1.2214190976	1.6339e-05	5.5504e-4
0.4	1.4918246	1.4918386304	1.3933e-	5.3977e-4

			05	
0.6	1.8221188	1.8221079424	1.0858e-05	2.8487e-4
0.8	2.2255409	2.2254971136	4.3815e-05	6.4294e-4
1	2.7182818	2.7182	8.1828e-05	2.1834e-4

Figure 2.1: Plot of comparison between the exact and approximate solutions of Example 1 for m= 2

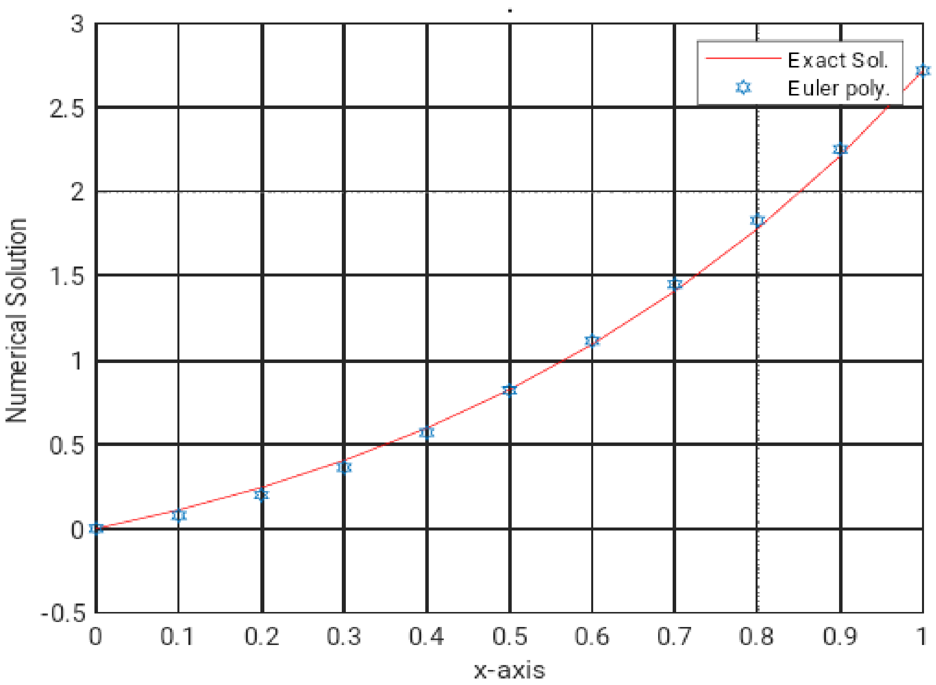


Figure 2.2: Plot of comparison between the exact and approximate solutions of Example 2 for m = 5

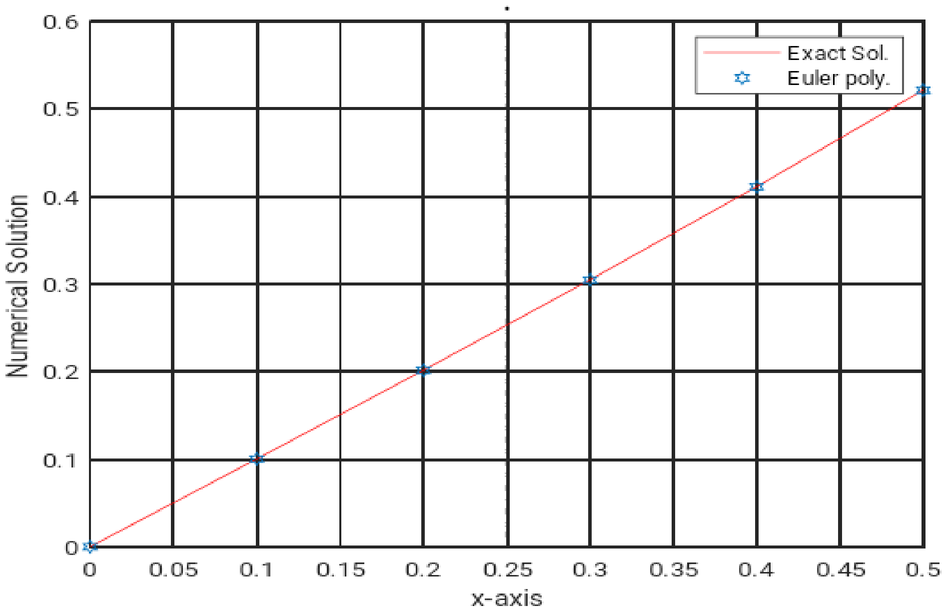
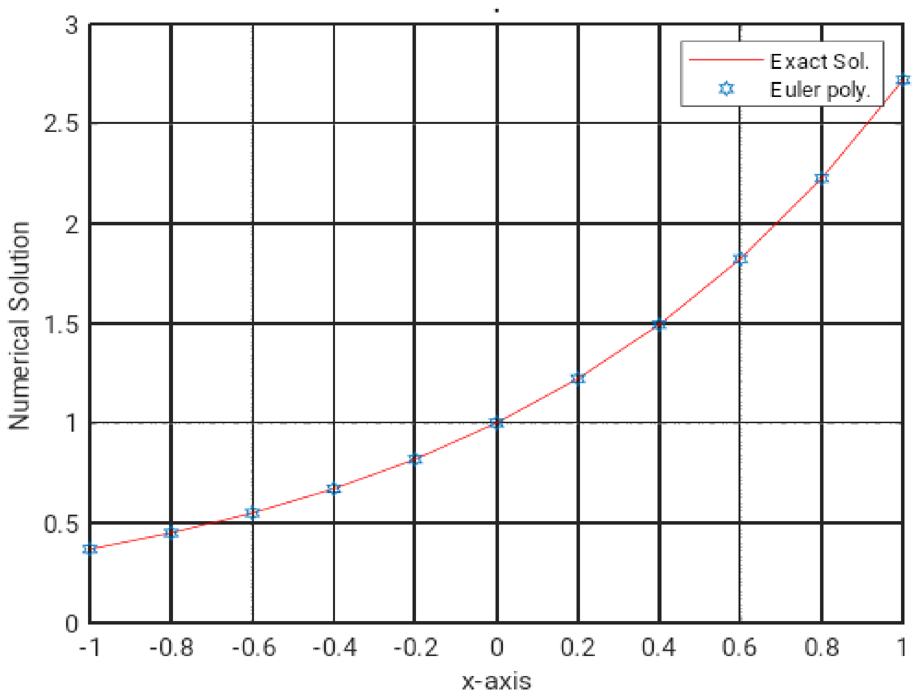


Figure 2.3: Plot of comparison between the exact and approximate solutions of Example 3 for m = 6



CONCLUSIONS

In this paper, we have introduced a new technique based on the combination of the least-squares method (LSM) with Euler polynomials for the approximate solutions of integro-differential equations subject to the mixed conditions. The solutions of first and second-orders linear FIDEs and VIDEs of the second type using the LSM method are considered polynomial as a basis function. We concluded from the figures and tables that the numerical results of a proposed method are accurate, efficient, and better than (AL-Juburee, 2010; Bildik et al, 2010; Yüzbaşı et al, 2011).

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