PalArch's Journal of Archaeology of Egypt / Egyptology

PROPERTIES OF ORTHOGONAL POLYNOMIAL MATRICES

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R. GAJALAKSHMI [1] AND G. RAMESH [2]. PROPERTIES OF ORTHOGONAL POLYNOMIAL MATRICES– Palarch's Journal Of Archaeology Of Egypt/Egyptology 17(12). ISSN 1567-214x

Keywords: Polynomial matrices, orthogonal matrices, symmetric matrices.

ABSTRACT
The concept of orthogonal polynomial matrices are introduced. Some properties and characterization for polynomial orthogonal matrices are obtained.

I. Introduction

In matrix theory, we come across some special types of matrices and two among them are symmetric matrix and orthogonal matrix. The term orthogonal matrix was used in 1854 by Charles Hermite in the Cambridge and Dublin Mathematical Journal, although it was not until 1878 that the formal definition of an orthogonal matrix was published by Frobenius. An orthogonal matrix is the real specialization of a unitary matrix and thus always a normal matrix. Orthogonal matrices are important for a number of reasons, both theoretical and practical.

A matrix $A(\lambda)$ is said to be a Polynomial matrix if all entries of $A(\lambda)$ are polynomials.Polynomials and polynomial matrices arise naturally as modeling tools in several areas of applied mathematics, science and engineering, especially in systems theory [2], [4], [5].

In this paper we have introduced polynomial orthogonal matrix and extend some properties of orthogonal matrices to orthogonal polynomial matrices.

I. Preliminaries

Definition 2.1 [1]

A matrix A is defined to be a rectangular arrangement of mn numbers arranged in m rows and n columns. If m = n the matrix is a square matrix. A^T is a matrix obtained from A by interchanging its rows into columns. If $A^T = A$ then A is said to be symmetric. A is said to be involutory if $A^2 = I$. It is orthogonal. If $AA^T = A^T A = I$. It is evident that orthogonal matrices are symmetric and determinant value of an orthogonal matrix is unity.

Definition 2.2 [2]

A matrix $A(\lambda)$ is said to be a polynomial matrix if all entries of $A(\lambda)$ are polynomials.

Definition 2.3 [3]

A square polynomial matrix $A(\lambda)$ is said to be symmetric if $A(\lambda) = A(\lambda)^T$ in other words all the coefficient matrices of $A(\lambda)$ are symmetric.
We state here some theorems without proof as they appeared earlier.

Definition 2.4 [1]
If A and B are two square orthogonal matrices, then
(i) AB is an orthogonal matrix. (ii) A+B is not an orthogonal matrix.
Theorem 2.5

If A is orthogonal then $\det A = \pm 1$.

III. Polynomial Orthogonal Matrix

Definition 3. 1

A polynomial orthogonal matrix is a polynomial matrix whose coefficient matrices are orthogonal.

Example 3. 2

Let

$$A(\lambda) = \begin{pmatrix} \frac{3}{5} - \frac{\lambda}{3} - \frac{7}{9}\lambda^2 & \frac{4}{5} + \frac{2}{3}\lambda + \frac{4}{9}\lambda^2 & \frac{2}{3}\lambda + \frac{4}{9}\lambda^2 \\ \frac{4}{5} + \frac{2}{3}\lambda + \frac{4}{9}\lambda^2 & -\frac{3}{5} - \frac{\lambda}{3} - \frac{1}{7}\lambda^2 & \frac{2}{3}\lambda + \frac{8}{9}\lambda^2 \\ \frac{2}{3}\lambda + \frac{4}{9}\lambda^2 & \frac{2}{3}\lambda + \frac{8}{9}\lambda^2 & 1 - \frac{\lambda}{3} - \frac{1}{7}\lambda^2 \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

Where $A_0 = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $A_1 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$,

$$A_2 = \frac{1}{9}\begin{pmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{pmatrix}$$

$$A_0 A_0^T = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A_0^T A_0 = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence $A_0 A_0^T = A_0^T A_0 = I$

$$A_1 A_1^T = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A_1^T A_1 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence $A_1 A_1^T = A_1^T A_1 = I$

$$A_2 A_2^T = \frac{1}{9}\begin{pmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{pmatrix}\frac{1}{9}\begin{pmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A_2^T A_2 = \frac{1}{9}\begin{pmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{pmatrix}\frac{1}{9}\begin{pmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence $A_2 A_2^T = A_2^T A_2 = I$

Theorem 3.3

A polynomial orthogonal matrix is always symmetric.

Proof

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_n\lambda^n$ be polynomial orthogonal matrix. Here coefficient matrix $A_i's$ are orthogonal matrices.

Since orthogonal matrices are symmetric, the coefficient matrices of $A(\lambda)$ are all symmetric.

Hence $A(\lambda)$ is a symmetric matrix.

Example 3.4

Consider the orthogonal polynomial matrix,

$$A(\lambda) = \begin{pmatrix} -1+\lambda^2 & -\lambda & 0 \\ -\lambda & -1+\lambda^2 & 0 \\ 0 & 0 & -1-\lambda+\lambda^2 \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2 .$$

Where $A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Here A_0, A_1, A_2 are orthogonal.

Also $A_0^T = A_0, A_1^T = A_1, A_2^T = A_2$.

That is A_0, A_1, A_2 all are symmetric.

Hence $A(\lambda)$ is symmetric.

Theorem 3.5

If $A(\lambda)$ is a $n \times n$ polynomial orthogonal matrix, then all of its coefficient

matrices are involutory.

Proof

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_n\lambda^n$ be polynomial orthogonal matrix. Here coefficient matrix $A_i's$ are orthogonal matrices.

That is, $A_i A_i^T = A_i^T A_i = I \qquad for\ i = 0,1,2.....n.$

Since orthogonal matrices are symmetric

That is, $A_i^T = A_i = I \qquad for\ i = 0,1,2.....n.$

Hence $A_i A_i^T = A_i\ A_i = A_i^2 = I \qquad for\ i = 0,1,2.....n.$

Hence each A_i is involutory.

Example 3.6

Let $A(\lambda) = \begin{pmatrix} \frac{3}{7}\lambda & -1-\frac{2}{7}\lambda & -\frac{6}{7}\lambda \\ -1-\frac{2}{7}\lambda & \frac{6}{7}\lambda & -\frac{3}{7}\lambda \\ -\frac{6}{7}\lambda & -\frac{3}{7}\lambda & -1-\frac{2}{7}\lambda \end{pmatrix} = A_0 + A_1\lambda$

Where $A_0 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $A_1 = \frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$

$$A_0^2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence $A_0^2 = I$

$$A_1^2 = \frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence $A_1^2 = I$

Hence all coefficient matrices A_0, A_1 are involutory .

Theorem 3.7

If $A(\lambda)$ is a polynomial orthogonal matrix if and only if $[A(\lambda)]^T$ is polynomial orthogonal matrix.

Proof

Let $A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_n\lambda^n$ be polynomial orthogonal

matrix. Here coefficient matrix $A_i's$ are orthogonal matrices. That is

$$\left.\begin{array}{l} A_0 A_0^T = A_0^T A_0 = I \\ A_1 A_1^T = A_1^T A_1 = I \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_n A_n^T = A_n^T A_n = I \end{array}\right\}. \qquad 3.1$$

To Prove $[A(\lambda)]^T$ is polynomial orthogonal matrix.

From 3. 1

$$[A(\lambda)]^T = A_0^T + A_1^T\lambda + A_2^T\lambda^2 \dots\dots\dots\dots\dots + A_n^T\lambda^n$$

We know that coefficient matrix $A_i's$ are orthogonal matrices.

Hence $[A(\lambda)]^T$ is polynomial orthogonal matrix.

Similarly we can prove the converse.

Example 3.8

Consider the orthogonal polynomial matrix,

$$A(\lambda) = \begin{pmatrix} \frac{3}{5}+\frac{30}{97}\lambda & \frac{4}{5}+\frac{97}{102}\lambda \\ \frac{4}{5}+\frac{97}{102}\lambda & -\frac{3}{5}-\frac{30}{97}\lambda \end{pmatrix} = A_0 + A_1\lambda$$

$$[A(\lambda)]^T = \begin{pmatrix} \frac{3}{5}+\frac{30}{97}\lambda & \frac{4}{5}+\frac{97}{102}\lambda \\ \frac{4}{5}+\frac{97}{102}\lambda & -\frac{3}{5}-\frac{30}{97}\lambda \end{pmatrix} = A_0 + A_1\lambda$$

Here $A(\lambda) = [A(\lambda)]^T$

Now $A(\lambda) = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} \frac{30}{97} & \frac{97}{102} \\ \frac{97}{102} & -\frac{30}{97} \end{pmatrix}\lambda$

Where $A_0 = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$, $A_1 = \begin{pmatrix} \frac{30}{97} & \frac{97}{102} \\ \frac{97}{102} & -\frac{30}{97} \end{pmatrix}$

$$A_0 A_0^T = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I .$$

Also $A_0 = A_0^T$.

$$A_1 A_1^T = \begin{pmatrix} \frac{30}{97} & \frac{97}{102} \\ \frac{97}{102} & -\frac{30}{97} \end{pmatrix}\begin{pmatrix} \frac{30}{97} & \frac{97}{102} \\ \frac{97}{102} & -\frac{30}{97} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Also $A_1 = A_1^T$.

Theorem 3.9

If $A(\lambda)$ is a polynomial orthogonal matrix then determinant of $A(\lambda)$ is a polynomial in λ.

Example 3.10

(i). Let $A(\lambda)=\begin{pmatrix} \frac{1}{\sqrt{2}}+\lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}+\lambda \end{pmatrix}=A_0+A_1\lambda$

$$\det A(\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix} 1+\sqrt{2}\lambda & 1 \\ 1 & -1+\sqrt{2}\lambda \end{pmatrix}$$

$$\det A(\lambda)=\frac{1}{\sqrt{2}}\left[(1+\sqrt{2}\lambda)(-1+\sqrt{2}\lambda)-1\right]$$

$$\det A(\lambda)=\frac{1}{\sqrt{2}}\left[-1+\sqrt{2}\lambda-\sqrt{2}\lambda+2\lambda^2-1\right]$$

$$\det A(\lambda)=\frac{1}{\sqrt{2}}\left[2\lambda^2-2\right]$$

$$\det A(\lambda)=\frac{2}{\sqrt{2}}\left[\lambda^2-1\right]=\sqrt{2}\left[\lambda^2-1\right]$$

Hence determinant of a polynomial orthogonal matrix is always polynomial.

(ii) Let $A(\lambda)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\lambda^0 \quad =A_0\lambda^0$

$$\det A(\lambda)=-1\neq 0$$

That is constant.

Hence determinant of a polynomial orthogonal matrices is a polynomial.

Remark 3.11

The product of two polynomial orthogonal matrices is not be orthogonal.

For, Let $A(\lambda)$ and $B(\lambda)$ be two polynomial orthogonal matrices.

Let $A(\lambda)=A_0+A_1\lambda+A_2\lambda^2+.....+A_n\lambda^n$ and

$B(\lambda)=B_0+B_1\lambda+B_2\lambda^2+.....+B_n\lambda^n$ be polynomial orthogonal matrices.

Where $A_0, A_1, A_2, ..., A_n$ and $B_0, B_1, B_2, ..., B_n$ are orthogonal matrices.

To prove $A(\lambda)$ and $B(\lambda)$ orthogonal.

$$A(\lambda)B(\lambda)=(A_0+A_1\lambda+A_2\lambda^2+.....+A_n\lambda^n)(B_0+B_1\lambda+B_2\lambda^2+.....+B_n\lambda^n)$$

$$=A_0B_0+(A_1B_0+B_1A_0)\lambda+.....+(A_0B_n+A_2B_{n-1}+.....+A_nB_0)\lambda^n$$

By the theorem (2.4) product of two orthogonal matrices are orthogonal.

So A_0B_0 is orthogonal matrix. But the second term of the above equation is not possible. Because sum of the two orthogonal matrices need not be orthogonal. Hence the product of two polynomial orthogonal matrices need not be orthogonal.

IV. Conclusion

Here we have extended some properties of orthogonal matrices to polynomial orthogonal matrices. All other properties can also be extended in a similar way.

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