PalArch's Journal of Archaeology of Egypt / Egyptology

# Fully-Discrete For Expanded $\mathrm{H}^1$-Galerkin Mixed Finite Element Method of PIDEs With Nonlinear Memory

Ali Kamil Naeemah [1], Hameeda Oda Al-Humedi [2]

[1,2] Department Mathematics, College of Education for Pure Sciences, Basrah University,

Basrah, Iraq

Email [1]: alimath1976@gmail.com , Email [2]: ahameeda722@yahoo.com

**Ali Kamil Naeemah, Hameeda Oda Al-Humedi. Fully-Discrete For Expanded $\mathrm{H}^1$-Galerkin Mixed Finite Element Method of PIDEs With Nonlinear Memory-- Palarch's Journal Of Archaeology Of Egypt/Egyptology x(x), xx-xx. ISSN 1567-214x**

**Keywords: $\mathrm{H}^1$-Galerkin mixed finite element method, fully-discrete, error estimates, parabolic integro-differential equations with nonlinear memory.**

**ABSTRACT**
In this paper, the expanded $H^1$-Galerkin mixed finite element method is proposed for parabolic integro-differential equations with nonlinear memory. The fully discrete error estimates based on backward Euler method are obtained. Moreover, the optimal a priori error estimates in $L^2$ and $H^1$-norm for the scalar unknown $u$ and the error results in $L^2$-norm for gradient $\boldsymbol{\sigma}$, and its flux $\boldsymbol{q}$ are derived. Finally, numerical results are presented to confirm our theoretical analysis.

## 1. INTRODUCTION
In this paper, we consider the following parabolic integro-differential equation with memory:

$$\begin{aligned} & u_t - \Delta u + \int_o^t K(t-s)\{-\nabla\cdot(\alpha(x,u)\nabla u + \beta(x,u) + \gamma(x,u)\cdot\nabla u + g(x,u))\} = f(x,t), \quad (x,t) \in \Omega\times J \\ & u(x,t) = 0, \qquad (x,t) \in \partial\Omega\times J, \qquad (1.1) \\ & u(x,0) = u_0(x), \qquad x \in \Omega, \end{aligned}$$

Where, $\Omega$ be a bounded polygonal domain $\mathbb{R}^d, (d = 1,2,3)$ with a smooth boundary $\partial\Omega$. $J = (0,T]$ is the time interval with $0 < T < \infty$, suppose that the kernel $k$ be positive definite as well as a smooth neither nonsmooth memory and $f$ is a known function. Let the function $\alpha(u)$ is a tensor function, $\beta(u)$ and $\gamma(u)$ are vector functions and $g(u)$ is scalar function. Then the functions $\alpha(u), \beta(u), \gamma(u)$ and $g(u)$ are continuously differentiable with respect to any variable also smooth and bounded. We consider that $\gamma(0) = 0$ and $g(0) = 0$.

Parabolic integro-differential equations with nonlinear memory are kind important of partial integro-differential equations, and have a many of applications in physical processes, such as the heat conduction and viscoelasticity in materials with memory[1, 2], to this type of equations, the approximating of solutions has been found by both finite element method in [3-5] finite deference method in [6] for the linear and nonlinear equations. recently, a lot of new numerical process such as discontinuous Galerkin method [7] finite volume element method[8], mixed finite element method[9], and a tow-grid method[10], have been proposed to solve PIDEs.

The traditional method has been studied and analyzed in [11] for parabolic integro-differential problems, in this method, the LBB stability condition must be met for the mixed finite element methods, which the select the choice of finite element spaces, in recent year, this problem was overcome by suggesting $H^1$-Galerkin mixed finite element method in [12] for partial parabolic differential equations. Also in [13] Z. Zhuo et al proposed $H^1$-Galerkin mixed finite element method with $H^1$-Galerkin expanded mixed finite element method in one method for approximate nonlinear hyperbolic equations of second order. The above method have a positive advantage, it is can direct solve both the scalar unknown, its gradient and its flux. Also it suitable for the case when the coefficient of the differential is a small tensor and does not need to be inverted. as well, continuous and piecewise (linear and higher-order) polynomials it is allowed to use in these formats in contrast to continuously differentiable piecewise polynomials required by standard $H^1$-Galerkin methods, and is free of LBB condition as required by the mixed finite element methods. Certainly, this formulation has its own disadvantages such as it needs to deal with the large size matrix.

The aim of this paper and on based [14] and [15] we will time discretization by Expanded $H^1$-Galerkin MFEM for parabolic integro-differential equation with nonlinear memory.

In this paper, we establish a fully-discrete for expanded $H^1$-Galerkin MFEM to the equation (1.1). Here we introduce as follows:

$\boldsymbol{q} = \nabla u - \int_0^t k(t-s)\big(\alpha(x,u)\nabla u + \beta(x,u)\big)ds$, and $\boldsymbol{\sigma} = \nabla u$.

For the time discretization, we consider the backward Euler method. An error estimates for the unknown function, gradient function, and flux in $L^2$-norms and $H^1$-norms are obtained.

Throughout this research, $C$ will denote a generic positive constant which does not depend on the spatial mesh parameter $h$ . and time discretization parameter $\Delta t$, and $\varepsilon$ denotes an arbitrarily small positive constant. we indicate to the natural inner product in $L^2(\Omega)$ or $\big(L^2(\Omega)\big)^2$ by $(\cdot,\cdot)$ with norm $\|\cdot\|_{L^2(\Omega)}$, In [16, 17] the other notations and definitions are used of Sobolev spaces. Let $X$ be a Banach space and $\psi(t)$: $[0,T] \longmapsto X$, we set

$$\|\psi(t)\|^2_{L^2(X)} = \int_0^{t_n-1} \|\psi(t)\|^2_{L^2(X)} ds, \qquad \|\psi(t)\|_{L^\infty(X)} = ess \sup_{0\le t\le T} \|\psi\|_X$$

## 2. The Weak Formulation

For simplicity of notation, we put $\alpha(x,u) = \alpha(u)$ , $\beta(x,u) = \beta(u)$, $\gamma(x,u) = \gamma(u)$, $g(x,u) = g(u)$, $\boldsymbol{q} = \nabla u - \int_0^t k(t-s)\big(\alpha(x,u)\nabla u + \beta(x,u)\big)ds$, and $\boldsymbol{\sigma} = \nabla u$, then equation (1.1) we can rewritten as

$$\left.\begin{aligned}
&(a)\quad u_t - \nabla\cdot\boldsymbol{q} + \int_0^t k(t-s)\big(\gamma(u)\cdot\boldsymbol{\sigma} + g(u)\big)ds = f,\\
&(b)\quad \boldsymbol{\sigma} = \nabla u,\\
&(c)\quad \boldsymbol{q} = \boldsymbol{\sigma} - \int_0^t k(t-s)\big(\alpha(u)\boldsymbol{\sigma} + \beta(u)\big)ds,\\
&(d)\quad u(x,0) = u_0(x).
\end{aligned}\right\} \quad (2.1)$$

After that weak form of the above equations is find $(u,\boldsymbol{\sigma},\boldsymbol{q}) \in H_0^1(\Omega)\times \boldsymbol{W}\times\boldsymbol{W}$ such that

$$\left.\begin{aligned}
&(a)\ (\boldsymbol{\sigma}_t,\boldsymbol{p}) + (\nabla\cdot\boldsymbol{q},\nabla\cdot\boldsymbol{p}) - \left(\int_0^t k(t-s)\big(\gamma(u).\boldsymbol{\sigma}+g(u)\big)ds,\nabla\cdot\boldsymbol{p}\right) = (f,\nabla\cdot\boldsymbol{p}),\\
&(b)\ (\boldsymbol{\sigma},\nabla v) = (\nabla u,\nabla v),\\
&(c)\ (\boldsymbol{q},\boldsymbol{w}) = (\boldsymbol{\sigma},\boldsymbol{w}) - \left(\int_0^t k(t-s)\big(\alpha(u)\boldsymbol{\sigma}+\beta(u)\big)ds,\boldsymbol{w}\right),\\
&(d)\ \boldsymbol{\sigma}(0) = \nabla u_0(x),
\end{aligned}\right\}\quad(2.2)$$

where, $\forall\ \boldsymbol{p}\in\boldsymbol{W},\ v\in V$ and

$$\boldsymbol{W} = H(div,\Omega) = \left\{\boldsymbol{w}\in\left(L^2(\Omega)\right)^d : \nabla\cdot\boldsymbol{w}\in L^2(\Omega)\right\},$$
$$V = H_0^1(\Omega) = \{v\in H^1(\Omega): v=0 \quad 0n \quad \partial\Omega\}.$$

## 3. The Fully-Discrete Expanded $H^1$-Galerkin Mixed Finite Element Method Scheme (MFEM)

Let $V_h, W_h$ are be finite dimensional subspaces of $H_0^1(\Omega)$ and $H(div,\Omega)$, respectively, which satisfy the following approximation properties [18, 19]:

$$\left.\begin{aligned}
&\inf_{v_h\in V_h} \| v-v_h\| + h\| v-v_h\|_1 \le ch^{m+1}\| v\|_{m+1}, && v\in H^{m+1}(\Omega),\\
&\inf_{p_h\in W_h} \|\boldsymbol{p}-\boldsymbol{p}_h\| \le ch^{k+1}\| p\|_{k+1}, && \boldsymbol{p}_h\in\left(H^{k+1}(\Omega)\right)^d,\\
&\inf_{p_h\in W_h} \|\nabla.\left(\boldsymbol{p}-\boldsymbol{p}_h\right)\| \le ch^{k+1}\| p\|_{k+1,} && \boldsymbol{p}_h\in\left(H^{k+1}(\Omega)\right)^d,
\end{aligned}\right\}\quad(3.1)$$

where $m,k$ are integers.

For the time discretization, we look the backward Euler method, let $0=t^0<t^1<\cdots< t^N=T$, be a given partition of the time interval $[0,T]$ with step length $\Delta t=\frac{T}{N}$, for some positive integer $N$. Define $t^n=n\Delta t$, $\phi=(\phi^n)$, $\bar{\partial}_t\phi^n=(\phi^n-\phi^{n-1})/\Delta t$ for smooth function $\phi$. To approximate the integral, we introduce the right rectangle quadrature rule.

$$q^n(\phi)=\Delta t\sum_{j=0}^{n-1} k_{n-j}\,\phi^j \approx \int_0^{t_n} k(t_n-s)\phi(s)ds, \tag{3.2}$$

where $k_{n-j}=k(t_n-s)$. The quadrature error

$$R^n(\phi)=q^n(\phi)-\int_0^{t_n} k(t_n-s)\phi(s)ds, \tag{3.3}$$

holds

$$|R^n(\phi)|\le C\Delta t\int_0^{t_n}(|\phi(s)|+|\phi_t(s)|)ds, \tag{3.4}$$

Where $k,\phi\in C^1[0,T]$.

The formula equivalent to the weak formula (2.2) we can write the following:

$$\left.\begin{aligned}
&(a)\quad (\partial_t \boldsymbol{\sigma}^n, \boldsymbol{p}) + (\nabla\cdot \boldsymbol{q}^n, \nabla\cdot \boldsymbol{p}) - \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \gamma\left(u(t_j)\right)\cdot \boldsymbol{\sigma}^j, \nabla\cdot \boldsymbol{p}\right)\\
&\quad -\left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, g\left(u(t_j)\right), \nabla\cdot \boldsymbol{p}\right) = (f^n, \nabla\cdot \boldsymbol{p}) + (R_1^n + R_2^n + R_3^n, \nabla\cdot \boldsymbol{p}),\\
&(b)\quad (\boldsymbol{\sigma}^n, \nabla v) = (\nabla u^n, \nabla v),\\
&(c)\quad (\boldsymbol{q}^n, \boldsymbol{w}) = (\boldsymbol{\sigma}^n, \boldsymbol{w}) - \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \alpha\left(u(t_j)\right) \boldsymbol{\sigma}^j, \boldsymbol{w}\right)\\
&\quad -\left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \beta\left(u(t_j)\right), \boldsymbol{w}\right) + (R_4^n + R_5^n, \boldsymbol{w}),
\end{aligned}\right\} \quad (3.5)$$

where $\forall \boldsymbol{p} \in \boldsymbol{W}, \forall v \in V$ and

$$R_1^n = \partial_t \sigma^n - \sigma_t = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (t_n - s)\sigma_{tt}\, ds,$$

$$R_2^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, \gamma\left(u(t_j)\right)\cdot \boldsymbol{\sigma}^j - \int_0^{t_n} k(t-s)\gamma(u)\cdot \boldsymbol{\sigma}(\tau)\, d\tau = C\Delta t \int_0^{t_n} (|\boldsymbol{\sigma}(\tau)| + |\boldsymbol{\sigma}_t(\tau)|)\, d\tau,$$

$$R_3^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, g\left(u(t_j)\right) - \int_0^{t_n} k(t-s) g(u)\, d\tau,$$

$$R_4^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, \alpha\left(u(t_j)\right) \boldsymbol{\sigma}^j - \int_0^{t_n} k(t-s)\alpha(u)\boldsymbol{\sigma}(\tau)\, d\tau = C\Delta t \int_0^{t_n} (|\boldsymbol{\sigma}(\tau)| + |\boldsymbol{\sigma}_t(\tau)|)\, d\tau,$$

$$R_5^n = \Delta t \sum_{j=0}^{n-1} k_{n-j}\, \beta\left(u(t_j)\right) - \int_0^{t_n} k(t-s)\beta(u)\, d\tau.$$

Then, we give a fully discrete method: find $(u_h^n, \boldsymbol{\sigma}_h^n, \boldsymbol{q}_h^n) \in V_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h, (n = 0,1,2,\dots, M-1)$, such that

$$\left.\begin{aligned}
&(a)\ (\partial_t \boldsymbol{\sigma}_h^n, \boldsymbol{p_h}) + (\nabla\cdot \boldsymbol{q}_h^n, \nabla\cdot \boldsymbol{p_h}) = \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \gamma\left(u_h(t_j)\right)\cdot \boldsymbol{\sigma}_h^j, \nabla\cdot \boldsymbol{p_h}\right)\\
&\qquad +\left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, g\left(u_h(t_j)\right), \nabla\cdot \boldsymbol{p_h}\right) + (f^n, \nabla\cdot \boldsymbol{p_h}),\\
&(b)\ (\boldsymbol{\sigma}_{\boldsymbol{h}}^n, \nabla v_h) = (\nabla u_h^n, \nabla v_h),\\
&(c)\ (\boldsymbol{q}_{\boldsymbol{h}}^n, \boldsymbol{w_h}) = (\boldsymbol{\sigma}_{\boldsymbol{h}}^n, \boldsymbol{w_h}) - \left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \alpha\left(u_h(t_j)\right) \boldsymbol{\sigma}_h^j, \boldsymbol{w_h}\right)\\
&\qquad -\left(\Delta t \sum_{j=0}^{n-1} k_{n-j}\, \beta\left(u_h(t_j)\right), \boldsymbol{w_h}\right),
\end{aligned}\right\} \quad (4.5)$$

where $\forall \boldsymbol{p_h} \in \boldsymbol{W}_h$ , $v_h \in V_h$ and

To determine the required error estimates, we introduce the following projection operators.

(i) From [20], let $I_h: H_0^1(\Omega) \longrightarrow V_h$ be the Ritz projection defined by:

$$(\nabla(u - I_h u_h), \nabla v_h) = 0, \qquad \forall v_h \in V_h, \qquad (3.7)$$

which the following results hold:

$$\| u - I_h u \| + h \| \nabla(u - I_h u) \| \le C h^{m+1} \| u \|_{m+1} . \quad (3.8)$$

(ii) From [21], let $R_h: H(div,\Omega) \longrightarrow W_h$ be the Raviart–Thomas projection defined by:

$$(\nabla\cdot(\boldsymbol{q} - R_h\boldsymbol{q}), \nabla\cdot\boldsymbol{p}_h) = 0 , \qquad \forall \boldsymbol{p}_h \in \boldsymbol{W}_h, \quad (3.9)$$

we have the following approximation features:

$$\| \boldsymbol{q} - R_h\boldsymbol{q} \| \le C h^{k+1} \| \boldsymbol{q} \|_{k+1} , \quad (3.10)$$

## 4. Convergence Analysis

Let

$$u(t_n) - u_h^n = u(t_n) - I_h u(t_n) + I_h u(t_n) - u_h^n = \delta^n + \varsigma^n$$
$$\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^n = \boldsymbol{\sigma}(t_n) - I_h\boldsymbol{\sigma}(t_n) + I_h\boldsymbol{\sigma}(t_n) - \boldsymbol{\sigma}_h^n = \boldsymbol{\eta}^n + \boldsymbol{\zeta}^n$$
$$\boldsymbol{q}(t_n) - \boldsymbol{q}_h^n = \boldsymbol{q}(t_n) - I_h\boldsymbol{q}(t_n) + I_h\boldsymbol{q}(t_n) - \boldsymbol{q}_h^n = \boldsymbol{\theta}^n + \boldsymbol{\xi}^n$$

Using (3.5) and (4.5) with help (3.7) and (3.9) at $t = t_n$, we get

$$\left.\begin{aligned}
&(a)\ \left(\bar{\partial}_t\boldsymbol{\zeta}^n, \boldsymbol{p}_h\right) + \left(\nabla\cdot\boldsymbol{\xi}^n, \nabla\cdot\boldsymbol{p}_h\right) = \left(\bar{\partial}_t\boldsymbol{\eta}^n, \boldsymbol{p}_h\right) + \left(R_1^n + R_2^n + R_3^n, \nabla\cdot\boldsymbol{p}_h\right)\\
&\qquad + \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j, \nabla\cdot\boldsymbol{p}_h\right)\\
&\quad + \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(\gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\eta}^j, \nabla\cdot\boldsymbol{p}_h\right) + \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j, \nabla\cdot\boldsymbol{p}_h\right)\\
&\qquad + \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(g\left(u(t_j)\right) - g\left(u_h(t_j)\right)\right), \nabla\cdot\boldsymbol{p}_h\right),\\
&(b)\ \left(\varsigma^n, \nabla v_h\right) = \left(\nabla\boldsymbol{\zeta}^n, \nabla v_h\right) + \left(\boldsymbol{\eta}^n, \nabla v_h\right),\\
&(c)\ \left(\boldsymbol{\xi}^n, \boldsymbol{w_h}\right) - \left(\boldsymbol{\zeta}^n, \boldsymbol{w_h}\right) = \left(\boldsymbol{\eta}^n, \boldsymbol{w_h}\right) - \left(\boldsymbol{\theta}^n, \boldsymbol{w_h}\right)\\
&\qquad - \left(\Delta t \textstyle\sum_{j=0}^{n-1} k_{n-j}\left(\alpha\left(u(t_j)\right) - \alpha\left(u_h(t_j)\right)\right)\boldsymbol{\sigma}^j, \boldsymbol{w}_h\right)\\
&\qquad - \left(\Delta t \textstyle\sum_{j=0}^{n-1} k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\eta}^j, \boldsymbol{w}_h\right) - \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j, \boldsymbol{w}_h\right)\\
&\qquad + \left(\Delta t \textstyle\sum_{j=1}^{n-1} k_{n-j}\left(\beta\left(u(t_j)\right) - \beta\left(u_h(t_j)\right)\right), \boldsymbol{w}_h\right) - \left(R_4^n + R_5^n, \boldsymbol{w}_h\right).
\end{aligned}\right\}(4.1)$$

Where $\forall \boldsymbol{p_h} \in W_h$, $v_h \in V_h$. Note that

$$\begin{aligned}
&\left(\gamma\left(u(t_j)\right)\cdot\boldsymbol{\sigma}^j - \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}_h^j\right)\\
&\quad = \gamma\left(u(t_j)\right)\cdot\boldsymbol{\sigma}^j - \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}^j + \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}^j - \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}_h^j\\
&\quad = \left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j + \gamma\left(u_h(t_j)\right)\cdot\left(\boldsymbol{\sigma}^j - \boldsymbol{\sigma}_h^j\right)\\
&\quad = \left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j + \gamma\left(u_h(t_j)\right)\cdot\left(\boldsymbol{\sigma}^j - I_h\boldsymbol{\sigma}^j + I_h\boldsymbol{\sigma}^j - \boldsymbol{\sigma}_h^j\right)\\
&\quad = \left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j + \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\eta}^j + \gamma\left(u_h(t_j)\right)\cdot\boldsymbol{\zeta}^j,
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left(\alpha\left(u(t_j)\right)\cdot\boldsymbol{\sigma}^j - \alpha\left(u_h(t_j)\right)\cdot\boldsymbol{\sigma}_h^j\right)\\
&\quad = \left(\alpha\left(u(t_j)\right) - \alpha\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j + \alpha\left(u_h(t_j)\right)\cdot\boldsymbol{\eta}^j + \alpha\left(u_h(t_j)\right)\cdot\boldsymbol{\zeta}^j,
\end{aligned}$$

**Theorem 4.1.** suppose that $u_h^0 = I_h(0)$, and $0 \le J \le N$. Then there exists a positive constant $C$ independent of $h$ and $\Delta t$ such that for $j = 0,1$ the following estimate holds

$$\begin{aligned}
\left\|u(t_J) - u_h^J\right\|_j + \left\|\boldsymbol{q}(t_J) - \boldsymbol{q}_h^J\right\|_j + \left\|\boldsymbol{\sigma}(t_J) - \boldsymbol{\sigma}_h^J\right\|_j &\le C h^{min(m+1,k+1)}\Big(\|u\|_{L^\infty(H^{m+1})} + \|q\|_{L^\infty(H^{k+1})}\\
&\quad + \|\sigma\|_{L^\infty(H^{k+1})} + \|\sigma_t\|_{L^\infty(H^{k+1})}\Big) + C\Delta t\left(\|u\|_{L^2(H^1)} + \|u_t\|_{L^2(H^1)} + \|u_{tt}\|_{L^2(H^1)}\right).
\end{aligned}$$

**Proof.** The estimates of $\delta^n, \boldsymbol{\eta}^n$ and $\boldsymbol{\theta}^n$ are given in (3.8) and (3,10) at $t = t_n$, it is enough to bound $\varsigma^n, \boldsymbol{\zeta}^n$and $\boldsymbol{\xi}^n$. We choose $v_h = \varsigma^n$ in (4.1(b)) to have

$$(\nabla\varsigma^n, \nabla\varsigma^n) = (\boldsymbol{\zeta}^n, \nabla\varsigma^n) + (\boldsymbol{\eta}^n, \nabla\varsigma^n), \tag{4.2}$$

Applying Young's inequalities on every term from terms to the above equation,

$$c\|\nabla\varsigma^n\|^2 + \varepsilon\|\nabla\varsigma^n\|^2 \le c\|\boldsymbol{\zeta}^n\|^2 + \varepsilon\|\nabla\varsigma^n\|^2 + c\|\boldsymbol{\eta}^n\|^2 + \varepsilon\|\nabla\varsigma^n\|^2 \tag{4.3}$$

Then

$$\|\nabla\varsigma^n\|^2 \le C(\|\boldsymbol{\eta}^n\|^2 + \|\boldsymbol{\zeta}^n\|^2) \tag{4.4}$$

Since $\varsigma^n \in V_h \subset H_0^1(\Omega)$, then, $\|\varsigma^n\| \le c_0\|\nabla\varsigma^n\|$, thus we can get the estimate $\|\varsigma^n\|$

$$\|\varsigma^n\|^2 \le C(\|\boldsymbol{\eta}^n\|^2 + \|\boldsymbol{\zeta}^n\|^2). \tag{4.5}$$

Here we estimate $\boldsymbol{\zeta}^n$, taking $\boldsymbol{p}_h = \boldsymbol{\zeta}^n$ in (4.1(a)) to obtain

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}_t\|\boldsymbol{\zeta}^n\|^2 + (\nabla\cdot\boldsymbol{\xi}^n, \nabla\cdot\boldsymbol{\zeta}^n) = (\partial_t\boldsymbol{\eta}^n, \boldsymbol{\zeta}^n) + (R_1^n, \nabla\cdot\boldsymbol{\zeta}^n) + (R_2^n, \nabla\cdot\boldsymbol{\zeta}^n) + (R_3^n, \nabla\cdot\boldsymbol{\zeta}^n),\\
&+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j, \nabla\cdot\boldsymbol{\zeta}^n\right) + \left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\eta}^j, \nabla\cdot\boldsymbol{\zeta}^n\right)\\
&+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j, \nabla\cdot\boldsymbol{\zeta}^n\right) + \left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(g\left(u(t_j)\right) - g\left(u_h(t_j)\right)\right), \nabla\cdot\boldsymbol{\zeta}^n\right)
\end{aligned} \tag{4.6}$$

use the Cauchy–Schwarz inequality and Young's inequality to every term ,then

$$|(\nabla\cdot\boldsymbol{\xi}^n, \nabla\cdot\boldsymbol{\zeta}^n)| \le \|\nabla\cdot\boldsymbol{\xi}^n\|\|\nabla\cdot\boldsymbol{\zeta}^n\| \le c\|\nabla\cdot\boldsymbol{\xi}^n\|^2 + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \tag{4.7}$$

$$|(\partial_t\boldsymbol{\eta}^n, \boldsymbol{\zeta}^n)| \le \|\partial_t\boldsymbol{\eta}^n\|\|\boldsymbol{\zeta}^n\| \le c\|\partial_t\boldsymbol{\eta}^n\|^2 + \varepsilon\|\boldsymbol{\zeta}^n\|^2. \tag{4.8}$$

$$|(R_1^n, \nabla\cdot\boldsymbol{\zeta}^n)| \le \|R_1^n\|\|\nabla\cdot\boldsymbol{\zeta}^n\| \le c\|R_1^n\|^2 + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2 . \tag{4.9}$$

$$|(R_2^n, \nabla\cdot\boldsymbol{\zeta}^n)| \le \|R_2^n\|\|\nabla\cdot\boldsymbol{\zeta}^n\| \le c\|R_2^n\|^2 + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \tag{4.10}$$

$$|(R_3^n, \nabla\cdot\boldsymbol{\zeta}^n)| \le \|R_3^n\|\|\nabla\cdot\boldsymbol{\zeta}^n\| \le c\|R_3^n\|^2 + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \tag{4.11}$$

$$\begin{aligned}
\left|\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j, \nabla\cdot\boldsymbol{\zeta}^n\right)\right| &\le c\left\|\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u(t_j)\right) - \gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\sigma}^j\right\|\|\nabla\cdot\boldsymbol{\zeta}^n\|\\
&\le cc_1c_2\left(\Delta t\sum_{j=1}^{n-1}\left\|\delta^j\right\|^2 + \left\|\varsigma^j\right\|^2\right) + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2 .
\end{aligned} \tag{4.12}$$

Where $c_1$ depends on $\|\boldsymbol{\sigma}^j\|$ and $c_2$ depends on $k_{n-j}$, and since $\gamma(u)$ is Lipchitz continuous with respect to $u$ then $\gamma\left(u(t_j)\right) = u(t_j)$ also $\gamma\left(u_h(t_j)\right) = u_h(t_j)$

$$\begin{aligned}
\left|\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\eta}^j, \nabla\cdot\boldsymbol{\zeta}^n\right)\right| &\le c\left\|\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\gamma\left(u_h(t_j)\right)\right)\cdot\boldsymbol{\eta}^j\right\|\|\nabla\cdot\boldsymbol{\zeta}^n\|\\
&\le cc_2c_3\left(\Delta t\sum_{j=1}^{n-1}\left\|\eta^j\right\|^2\right) + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2 .
\end{aligned} \tag{4.13}$$

Where $c_3$ depends on $\gamma\left(u_h(t_j)\right)$,

$$\begin{aligned}
\left|\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j, \nabla\cdot\boldsymbol{\zeta}^n\right)\right| &\le c\left\|\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j\right\|\|\nabla\cdot\boldsymbol{\zeta}^n\|\\
&\le cc_2c_4\left(\Delta t\sum_{j=1}^{n-1}\|\boldsymbol{\zeta}^j\|^2\right) + \varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2 .
\end{aligned} \tag{4.14}$$

Where $c_4$ depends on $\alpha\left(u_h(t_j)\right)$,

$$\left|\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(g\left(u(t_j)\right)-g\left(u_h(t_j)\right)\right),\nabla\cdot\boldsymbol{\zeta}^n\right)\right|\le cc_2\left\|\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(g\left(u(t_j)\right)-g\left(u_h(t_j)\right)\right)\right\|\|\nabla\cdot\boldsymbol{\zeta}^n\|$$
$$\le cc_2\left(\Delta t\sum_{j=1}^{n-1}(\|\delta^j\|^2+\|\varsigma^j\|^2)\right)+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \qquad (4.15)$$

Since $g(u)$ is Lipchitz continuous with respect to $u$ then $g\left(u(t_j)\right)=u(t_j)$ also $g\left(u_h(t_j)\right)=u_h(t_j)$.
Substituting (4.7)-(4.15) into (4.6) we get

$$\frac{1}{2}\bar{\partial}_t\|\boldsymbol{\zeta}^n\|^2+c\|\nabla\cdot\boldsymbol{\xi}^n\|^2+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2\le c\|\partial_t\boldsymbol{\eta}^n\|^2+\varepsilon\|\boldsymbol{\zeta}^n\|^2+c\|R_1^n\|^2+c\|R_2^n\|^2+c\|R_3^n\|^2$$
$$+cc_1c_2\left(\Delta t\sum_{j=1}^{n-1}(\|\delta^j\|^2+\|\varsigma^j\|^2)\right)+cc_2c_3\left(\Delta t\sum_{j=1}^{n-1}\|\eta^j\|^2\right)$$
$$+cc_2c_4\left(\Delta t\sum_{j=1}^{n-1}\|\boldsymbol{\zeta}^j\|^2\right)+cc_2\left(\Delta t\sum_{j=1}^{n-1}(\|\delta^j\|^2+\|\varsigma^j\|^2)\right)+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2 \qquad (4.16)$$

Then

$$\frac{1}{2}\bar{\partial}_t\|\boldsymbol{\zeta}^n\|^2+\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le C\left(\|\partial_t\boldsymbol{\eta}^n\|^2+\|R_1^n\|^2+\|R_2^n\|^2+\|R_3^n\|^2+\Delta t\sum_{j=1}^{n-1}\left(\|\delta^j\|^2+\|\varsigma^j\|^2\right)+\Delta t\sum_{j=1}^{n-1}\|\eta^j\|^2\right.$$
$$\left.+\Delta t\sum_{j=1}^{n-1}\|\boldsymbol{\zeta}^j\|^2\right)+\varepsilon\|\boldsymbol{\zeta}^n\|^2+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \qquad (4.17)$$

using Taylor formula we can derive

$$\|R_1^n\|^2\le C\Delta t\int_{t_{n-1}}^{t_n}\|\sigma_{tt}\|^2ds,\qquad \|\partial_t\boldsymbol{\eta}^n\|^2\le Ch^{2(k+1)}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\eta_t(s)\|^2\,ds,$$

$$\|R_2^n\|^2=\left\|\Delta t\sum_{j=0}^{n-1}k_{n-j}\,\gamma\left(u(t_j)\right)\cdot\boldsymbol{\sigma}^j-\int_0^{t_n}k(t-s)\gamma\left(u(t_j)\right)\cdot\boldsymbol{\sigma}(\tau)d\tau\right\|^2$$
$$\le C(\Delta t)^2\int_O^{t^n}\{\|u\|^2+\|u_t\|^2\}ds,$$

And

$$\|R_3^n\|^2=\left\|\Delta t\sum_{j=0}^{n-1}k_{n-j}\,g\left(u(t_j)\right)-\int_0^{t_n}k(t-s)g(u)d\tau\right\|^2$$
$$\le C(\Delta t)^2\int_O^{t^n}\{\|u\|^2+\|u_t\|^2\}ds,$$

Therefore, (4.17) becomes

$$\frac{1}{2}\bar{\partial}_t\|\boldsymbol{\zeta}^n\|^2+\left\|\nabla\cdot\boldsymbol{\xi}^n\right\|^2\le C\left(h^{k+1}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\sigma_t(s)\|^2\,ds+\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2ds+(\Delta t)^2\int_{O}^{t^n}\{\|u\|^2+\|u_t\|^2\}ds\right.$$
$$\left.+\Delta t\sum_{j=1}^{n-1}\left(\left\|\delta^j\right\|^2+\left\|\varsigma^j\right\|^2\right)+\Delta t\sum_{j=1}^{n-1}\left\|\eta^j\right\|^2+\Delta t\sum_{j=1}^{n-1}\left\|\boldsymbol{\zeta}^j\right\|^2\right)+\varepsilon\|\boldsymbol{\zeta}^n\|^2+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \qquad (4.18)$$

So that

$$\frac{\|\boldsymbol{\zeta}^n\|^2-\|\boldsymbol{\zeta}^{n-1}\|^2}{\Delta t}+2\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le C\left(h^{k+1}\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|\sigma_t(s)\|^2\,ds+\Delta t\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2ds\right.$$
$$\left.+(\Delta t)^2\int_{O}^{t^n}\{\|u\|^2+\|u_t\|^2\}ds+\Delta t\sum_{j=1}^{n-1}(\|\delta^j\|^2+\|\varsigma^j\|^2)+\Delta t\sum_{j=1}^{n-1}\|\eta^j\|^2+\Delta t\sum_{j=1}^{n-1}\|\boldsymbol{\zeta}^j\|^2\right)$$
$$+\varepsilon\|\boldsymbol{\zeta}^n\|^2+\varepsilon\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \qquad (4.19)$$

Then, multiplying by $\Delta t$ of the above equation, we get

$$\|\boldsymbol{\zeta}^n\|^2-\|\boldsymbol{\zeta}^{n-1}\|^2+2\Delta t\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le Ch^{k+1}\int_{t_{n-1}}^{t_n}\|\sigma_t(s)\|^2\,ds+C(\Delta t)^2\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2ds$$
$$+C(\Delta t)^2\int_{O}^{t^n}\{\|u\|^2+\|u_t\|^2\}ds+C(\Delta t)^2\sum_{j=1}^{n-1}(\|\delta^j\|^2+\|\varsigma^j\|^2+\|\eta^j\|^2+\|\boldsymbol{\zeta}^j\|^2)$$
$$+\varepsilon\Delta t\|\boldsymbol{\zeta}^n\|^2+\varepsilon\Delta t\|\nabla\cdot\boldsymbol{\zeta}^n\|^2. \qquad (4.20)$$

Summing from $n=1,2,\dots,J$, we obtain

$$\|\boldsymbol{\zeta}^n\|^2+2\Delta t\sum_{n=1}^{J}\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le\|\boldsymbol{\zeta}^0\|^2+Ch^{k+1}\|\sigma_t\|^2_{L^2(H^{k+1})}+C(\Delta t)^2\left(\|u\|^2_{L^2(H^1)}+\|u_t\|^2_{L^2(H^1)}\right.$$
$$\left.+\|u_{tt}\|^2_{L^2(H^1)}\right)+C(\Delta t)^2\sum_{n=1}^{J}\sum_{j=1}^{n-1}\left(\left\|\delta^j\right\|^2+\left\|\varsigma^j\right\|^2+\left\|\eta^j\right\|^2+\left\|\boldsymbol{\zeta}^j\right\|^2\right)+\varepsilon\Delta t\sum_{n=1}^{J}\left(\left\|\boldsymbol{\zeta}^n\right\|^2+\|\nabla\cdot\boldsymbol{\zeta}^n\|^2\right), \qquad (4.21)$$

Putting (4.5) into (4.21) then using discrete Gronwall's lemma with $\boldsymbol{\zeta}^0=0$, we get

$$\|\boldsymbol{\zeta}^n\|^2+2\Delta t\sum_{n=1}^{J}\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le Ch^{k+1}\|\sigma_t\|^2_{L^2(H^{k+1})}+C(\Delta t)^2\left(\|u\|^2_{L^2(H^1)}+\|u_t\|^2_{L^2(H^1)}\right.$$
$$\left.+\|u_{tt}\|^2_{L^2(H^1)}\right)+C(\Delta t)^2\sum_{n=1}^{J}\sum_{j=1}^{n-1}\left(\left\|\delta^j\right\|^2+\left\|\eta^j\right\|^2\right), \qquad (4.22)$$

after that, we obtain

$$\|\boldsymbol{\zeta}^n\|^2+2\Delta t\sum_{n=1}^{J}\|\nabla\cdot\boldsymbol{\xi}^n\|^2\le Ch^{k+1}\|\sigma_t\|^2_{L^2(H^{k+1})}+C(\Delta t)^2\left(\|u\|^2_{L^2(H^1)}+\|u_t\|^2_{L^2(H^1)}+\|u_{tt}\|^2_{L^2(H^1)}\right)$$
$$+C\left(h^{2(m+1)}\|u\|^2_{L^\infty(H^{m+1})}+h^{2(k+1)}\|\sigma\|^2_{L^\infty(H^{k+1})}\right), \qquad (4.23)$$

then

$$\|\boldsymbol{\zeta}^n\|+2\Delta t\sum_{n=2}^{J}\|\nabla\cdot\boldsymbol{\xi}^n\|\le C\,h^{min(m+1,k+1)}\left(\|u\|_{L^\infty(H^{m+1})}+\|\sigma\|_{L^\infty(H^{k+1})}+\|\sigma_t\|_{L^\infty(H^{k+1})}\right)$$
$$+C\Delta t\left(\|u\|_{L^2(H^1)}+\|u_t\|_{L^2(H^1)}+\|u_{tt}\|_{L^2(H^1)}\right), \qquad (4.24)$$

Substituting (4.24) into (4.5) to have

$$\|\varsigma^n\|\le Ch^{min(m+1,k+1)}\left(\|u\|_{L^\infty(H^{m+1})}+\|\sigma\|_{L^\infty(H^{k+1})}+\|\sigma_t\|_{L^\infty(H^{k+1})}\right)$$

$$+C\Delta t\left(\|u\|_{L^2(H^1)}+\|u_t\|_{L^2(H^1)}+\|u_{tt}\|_{L^2(H^1)}\right). \tag{4.25}$$

Choose $\boldsymbol{w_h}=\boldsymbol{\xi}^n$ in (4.1(c)) to get

$$(\boldsymbol{\xi}^n,\boldsymbol{\xi}^n)=(\boldsymbol{\eta}^n,\boldsymbol{\xi}^n)-(\boldsymbol{\zeta}^n,\boldsymbol{\xi}^n)-(\boldsymbol{\theta}^n,\boldsymbol{\xi}^n)-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(\alpha\left(u(t_j)\right)-\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\sigma}^j,\boldsymbol{\xi}^n\right)$$
$$-\left(\Delta t\sum_{j=0}^{n-1}k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\eta}^j,\boldsymbol{\xi}^n\right)-\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\alpha\left(u_h(t_j)\right)\right)\boldsymbol{\zeta}^j,\boldsymbol{\xi}^n\right)-(R_4^n+R_5^n,\boldsymbol{\xi}^n)$$
$$+\left(\Delta t\sum_{j=1}^{n-1}k_{n-j}\left(\beta\left(u(t_j)\right)-\beta\left(u_h(t_j)\right)\right),\boldsymbol{\xi}^n\right), \tag{4.26}$$

using Young's inequalities to (4.26), one has

$$\|\boldsymbol{\xi}^n\|^2\le C(\|\boldsymbol{\eta}^n\|^2+\|\boldsymbol{\zeta}^n\|^2+\|\boldsymbol{\theta}^n\|^2)+C\left(\Delta t\sum_{j=1}^{n-1}\|\delta^j\|^2+\|\varsigma^j\|^2+\|\eta^j\|^2+\|\zeta^j\|^2\right)$$
$$+C(\|R_4^n\|^2+\|R_5^n\|^2)+\varepsilon\|\boldsymbol{\xi}^n\|^2. \tag{4.27}$$

Where

$$\|R_4^n\|^2=\left\|\Delta t\sum_{j=0}^{n-1}k_{n-j}\,\alpha\left(u(t_j)\right)\boldsymbol{\sigma}^j-\int_0^{t_n}k(t-s)\alpha(u)\boldsymbol{\sigma}(\tau)d\tau\right\|^2$$
$$\le C(\Delta t)^2\int_O^{t^n}\{\|u\|^2+\|u_t\|^2\}ds,$$

and

$$\|R_5^n\|^2=\left\|\Delta t\sum_{j=0}^{n-1}k_{n-j}\,\beta\left(u(t_j)\right)-\int_0^{t_n}k(t-s)\beta(u)d\tau.\right\|^2$$
$$\le C(\Delta t)^2\int_O^{t^n}\{\|u\|^2+\|u_t\|^2\}ds,$$

Substituting the above inequalities into (4.27) one has

$$\|\boldsymbol{\xi}^n\|^2\le C(\|\boldsymbol{\eta}^n\|^2+\|\boldsymbol{\zeta}^n\|^2+\|\boldsymbol{\theta}^n\|^2)+C\left(\Delta t\sum_{j=1}^{n-1}\|\delta^j\|^2+\|\varsigma^j\|^2+\|\eta^j\|^2+\|\zeta^j\|^2\right)$$
$$+C(\Delta t)^2\int_O^{t^n}\{\|u\|^2+\|u_t\|^2\}ds+\varepsilon\|\boldsymbol{\xi}^n\|^2. \tag{4.28}$$

Setting (4.24) and (4.25) into (4.28) and applying Gronwall's lemma, we get

$$\|\xi^n\|\le Ch^{min(m+1,k+1)}\left(\|u\|_{L^\infty(H^{m+1})}+\|q\|_{L^\infty(H^{k+1})}+\|\sigma\|_{L^\infty(H^{k+1})}+\|\sigma_t\|_{L^\infty(H^{k+1})}\right)$$
$$+C\Delta t\left(\|u\|_{L^2(H^1)}+\|u_t\|_{L^2(H^1)}+\|u_{tt}\|_{L^2(H^1)}\right). \tag{4.29}$$

The use of the triangle inequality then (4.24), (4.25) and (4.29) with (3.8), (3.10) completes the proof.

**Remark:** $C=C(c_0,c_1,c_2,c_3,c_4,\Delta t)$ ■

**Table 1:** The errors of $u$, $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ .

| $(h,\Delta t)$ | $\| u-u_h \|_{L^\infty\left(L^2(\Omega)\right)}$ | $\| \boldsymbol{\sigma}-\boldsymbol{\sigma}_h \|_{L^\infty\left(L^2(\Omega)\right)}$ | $\| \boldsymbol{q}-\boldsymbol{q}_h \|_{L^\infty\left(L^2(\Omega)\right)}$ |
|---|---|---|---|
| $(1/4,1/2)$ | 0.0011 | 0.0100 | 0.0150 |

| | | | |
|---|---|---|---|
| $(1/8, 1/4)$ | 5.9374e-04 | 0.0040 | 0.0110 |
| $(1/16, 1/8)$ | 3.2062e-04 | 0.0020 | 0.0093 |
| $(1/32, 1/16)$ | 1.6768e-04 | 0.0010 | 0.0083 |

**Table 2:** The orders of convergence for $u, \boldsymbol{\sigma}$ and $\boldsymbol{q}$.

| $(h, \Delta t)$ | $\| u - u_h \|_{L^\infty(L^2(\Omega))}$ | $\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{L^\infty(L^2(\Omega))}$ | $\| \boldsymbol{q} - \boldsymbol{q}_h \|_{L^\infty(L^2(\Omega))}$ |
|---|---|---|---|
| $(1/4, 1/2)$ | 0.8896 | 1.3219 | 0.4475 |
| $(1/8, 1/4)$ | 0.8890 | 1.00 | 0.2422 |
| $(1/16, 1/8)$ | 0.9352 | 1.00 | 0.1644 |

## 5. Numerical Example

The aim of this section is to given a numerical example to illustrate our theoretical analysis results obtained in Section 4. We consider the exact solution $u$ for (1.1) is chosen as
$u(x_1, x_2; t) = x_1(1 - x_1)x_2(1 - x_2)e^{-t}$, and $u_0(x_1, x_2; t) = x_1(1 - x_1)x_2(1 - x_2)$
Where $\Omega = [0,1] \times [0,1]$, $J = (0, 2]$, $k(t - s) = e^{-(t-s)}$, $g(x, t) = \sin u$, $\alpha(x, t) = 0$, $\beta(x, t) = (\sin u,\ 1 - \cos u)^T$, $\gamma(x, t) = (1 - \cos u,\ \sin u)^T$, and

$$f(x_1, x_2; t) = (2x_1(1 - x_1) - x_1(1 - x_1)x_2(1 - x_2) + 2x_2(1 - x_2) + (1 - 2x_1)x_2(1 - x_2)t)e^{-t} - (1 - 2x_1)x_2(1 - x_2)e^{-t} \int_0^t cos(x_1(1 - x_1)x_2(1 - x_2)e^{-s})\,ds + e^{-t}\int_0^t e^s sin(x_1(1 - x_1)x_2(1 - x_2)e^{-s})\,ds$$

The domain $\Omega$ is divided into the triangulations with grid size $h_u = h_\sigma = h_q = h$ uniformly, also the time interval $[0, T]$ was divided into $N$ subintervals $0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$ with step length $\Delta t = T/N$ for some positive integer $N$. And considering the mixed finite element spaces $V_h$ which consists of linear polynomials for the scalar unknown function $u$ and the author space $W_h$ consists of linear polynomials for the gradient $\sigma$ and its flux $q$ and using the backward Euler procedure. We get some convergence results for $\| u - u_h \|_{L^\infty(L^2(\Omega))}$, $\| \sigma - \sigma_h \|_{L^\infty(L^2(\Omega))}$, $\| q - q_h \|_{L^\infty(L^2(\Omega))}$ with $h = 1/4, 1/8, 1/16, 1/32$ and $\Delta t = 1/2, 1/4, 1/8, 1/16$ in **Table 1**, and we obtain the orders of convergence in **Table 2**. The Figures in 1, 3, and 5 is shown exact solution of $u, \sigma, q$, respectively, and the corresponding Figures in 2, 4, and 6 is shown numerical solution $u_h, \sigma_h, q_h$ respectively.

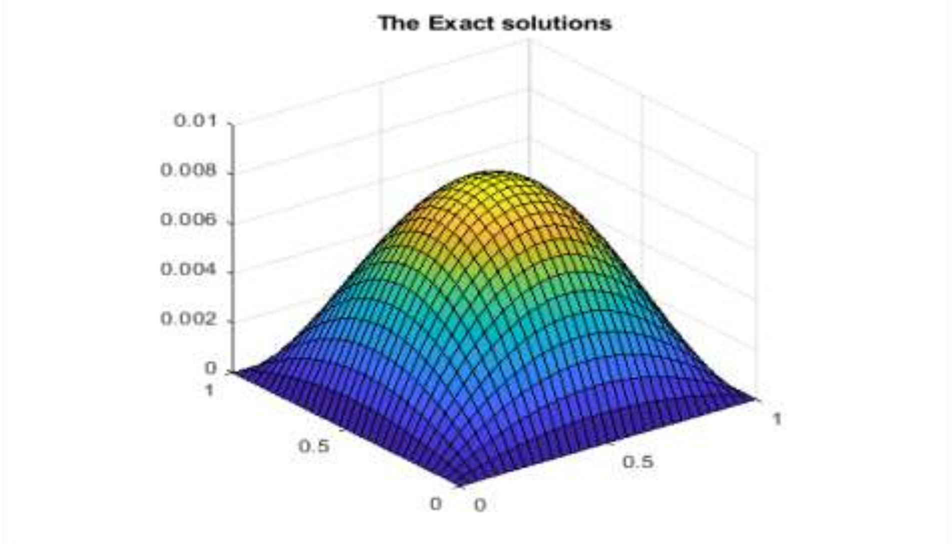



**Figure 1:** The exact solution of $u$ at $t = 2$.

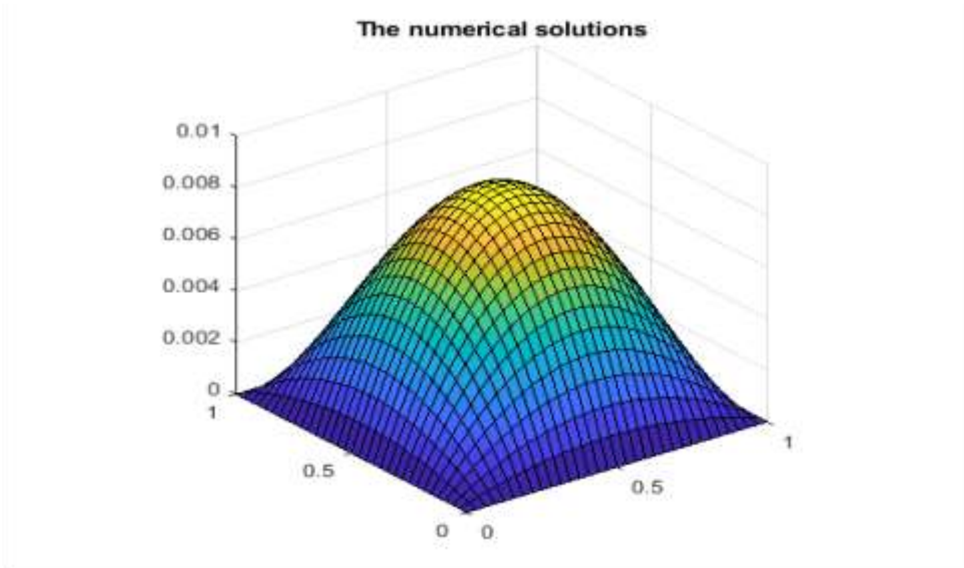


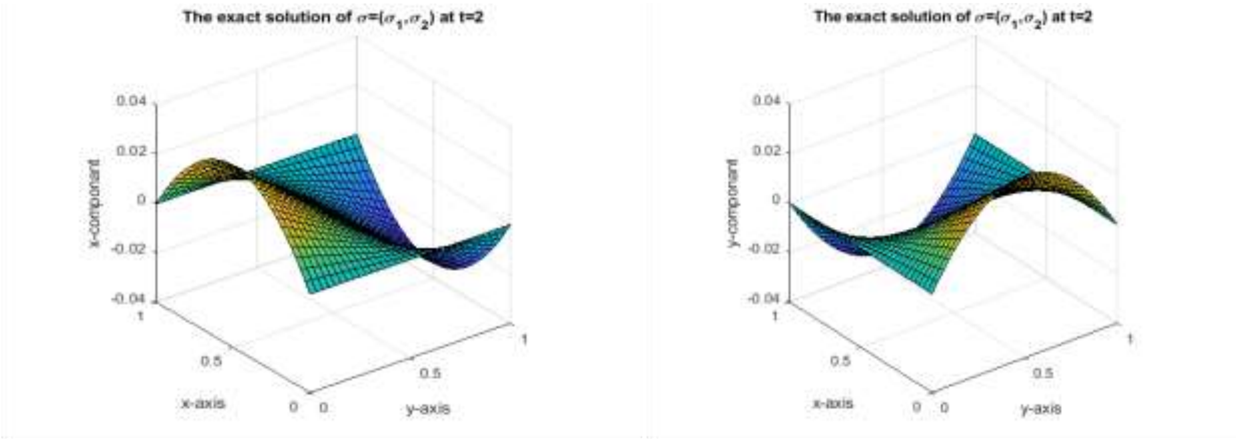

**Figure 2:** The numerical solution of $u_h$ at $t = 2$.

The corresponding exact gradient is

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = (x_1 x_2 e^{-t}(x_2 - 1) + x_1 x_2 e^{-t}(x_1 - 1)(x_2 - 1), \quad x_1 x_2 e^{-t}(x_1 - 1) + x_1 e^{-t}(x_1 - 1)(x_2 - 1)),$$

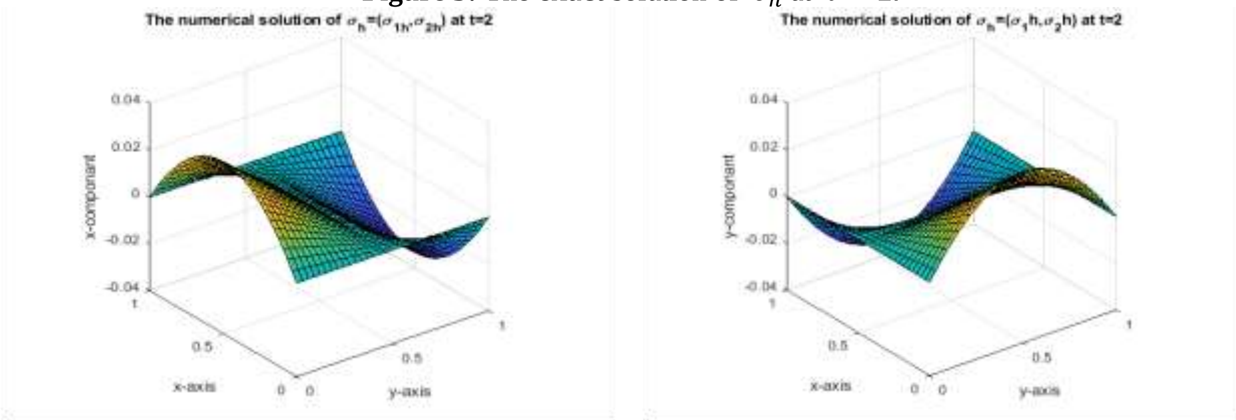
and its exact flux is

$$\boldsymbol{q} = (q_1, q_2) = (\sin(x_1 x_2 e^{-t}(x_1 - 1)(x_2 - 1))(e^{-t} - 1) \quad x_1 x_2 e^{-t}(x_2 - 1) + x_1 e^{-t}(x_1 - 1)(x_2 - 1)), (e^{-t} - 1) \quad -cos(x_1 x_2 e^{-t}(x_1 - 1)(x_2 - 1))(e^{-t} - 1) + x_1 x_2 e^{-t}(x_1 - 1) \quad + x_1 e^{-t}(x_1 - 1)(x_2 - 1)).$$



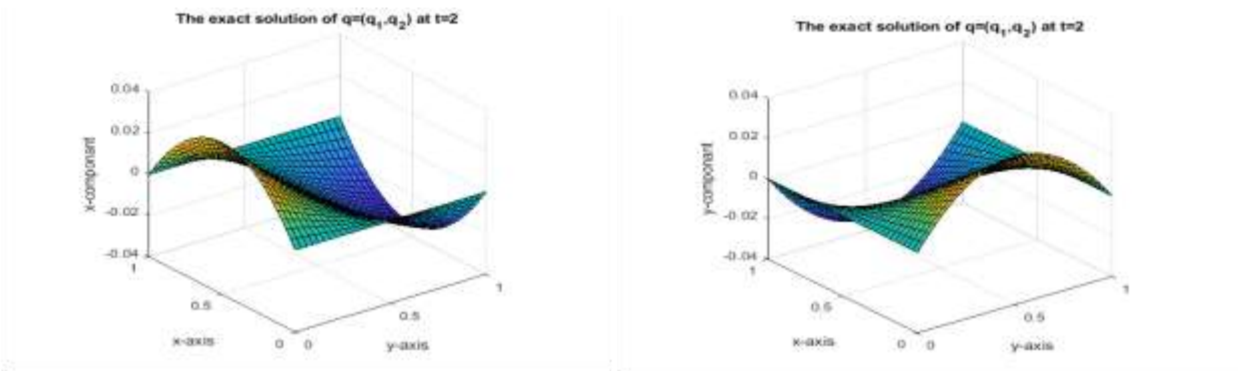

$(a)$ $(b)$

**Figure 3:** The exact solution of $\sigma_h$ at $t = 2$.



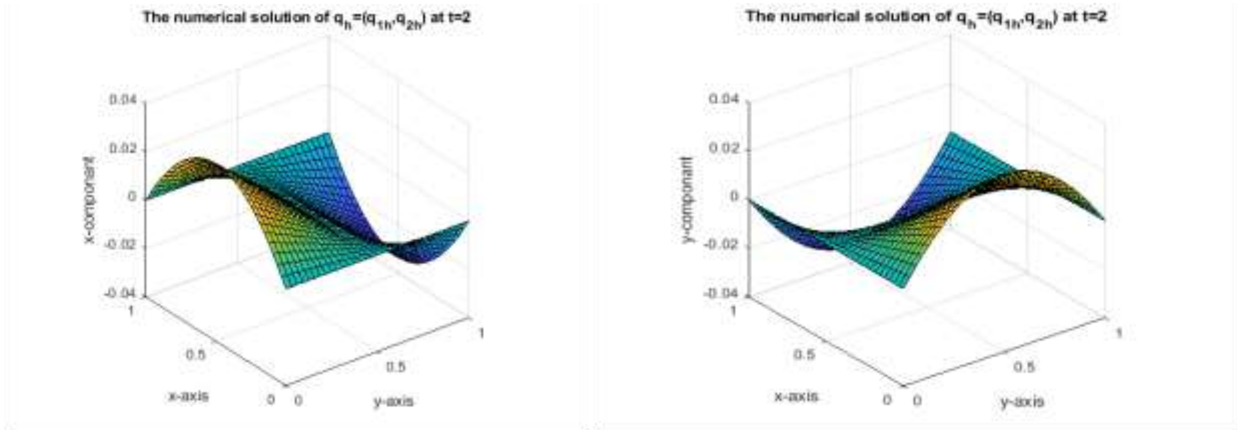

$(a)$ $(b)$

**Figure 4:** The numerical solution of $\sigma_h$ at $t = 2$.




$(a)$ $(b)$

**Figure 5:** The exact solution of $q_h$ at $t = 2$.




$(a)$ $(b)$

**Figure 6:** The numerical solution of $q_h$ at $t = 2$.

## 6. Conclusion

In this paper, expanded $H^1$-Galerkin Mixed Finite Element Method is discussed for parabolic integro-differential equations with nonlinear memory. This method could solve $u, \boldsymbol{\sigma}$ and $\boldsymbol{q}$ directly. The error estimates are derived fully discrete schemes. Certainly, the formulation has its own disadvantages such as it needs to deal with the large size matrix. Finally, some numerical results are provided to confirm our theoretical analysis.

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